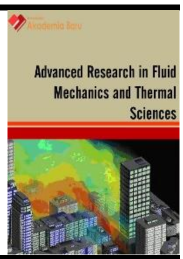
Penerbit Akademia Baru

Journal of Advanced Research in Fluid Mechanics and Thermal Sciences

Journal homepage: www.akademiabaru.com/arfmts.html
ISSN: 2289-7879



Convergence and Error Analysis of a Bi-quadratic Triangular Galerkin Finite Element Model for Heat Conduction Simulation

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ARTICLE INFO

Article history:
Received 13 October 2018
Received in revised form 2 December 2018
Accepted 23 January 2019
Available online 7 February 2019

ABSTRACT

Present paper elaborates solution of partial differential equations (PDE) of two dimensional steady state heat conduction by using a bi quadratic triangular Galerkin's finite element method (QGFEM). The steady state heat distribution is modeled by a two-dimensional Laplace partial differential equations. A six- point triangular planar finite element model is developed for the QGFEM based on quadratic basis functions on the Cartesian coordinate system where physical domain is meshed by structured grid. The elemental stiffness matrix is formulated by using a direct integration scheme along the trilateral domain area without the necessity to use the Jacobian matrix. Validation is conducted to an analytical solution of a rectangular plate having mixed, asymmetric boundary conditions. Comparisons of the present QGFEM results and the exact solution show promising results. The convergence of the method is presented by checking the error analysis for various number of elements used for the simulation.

Keywords:
Bi-quadratic triangular finite element model, Galerkin method, error analysis, heat conduction, partial differential equation

1. Introduction

Finite element method (FEM) is a widely used numerical technique to solve partial differential equations (PDE) arising in mathematical model of many engineering fields [1-9]. There are many approaches in the FEM techniques such as a direct stiffness method, minimum total potential energy method and residual methods [2]. In the residual method, the PDE solution is approximated by a basis function [9] where each basis function is selected such that it matches elemental boundary conditions. The basis function is substituted into the PDE model and arranged to give the residual function [10]. The Galerkin's finite element method (GFEM) uses a weighted residual method where

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it is formed based on the first derivative of the basic function with respect to the nodal variables [11]. The integral of the product is usually presented in a weak formulation by performing integration by part technique where the first derivative of the function is required. For this reason, GFEM is suitable since it uses the first derivative of the trial function.

The integration on the domain in FEM is usually performed by dividing the total domain into discrete subdomains or elements. The integration of each subdomain is conducted to construct the local stiffness matrix as well as the load vector matrix. The present work attempts to implement the Galerkin's weighted residual procedure above by using direct integration without the use of the Jacobian matrix. A simplified stiffness matrix is proposed that can be used for a heat conduction planar domain problem. The proposed scheme is suitable for a structured grid mesh generation where its scheme can reduce significantly the CPU time. The exact solution for this particular problem is derived and the results are compared to the GFEM results.

2. Mathematical Formulation

A steady state heat conduction/flow problem with no heat source in a homogeneous domain can be modeled as PDE in the form of Laplace's equation [8-11] which can be combined with inhomogeneous Dirichlet or Neumann conditions as shown in Figure 1:

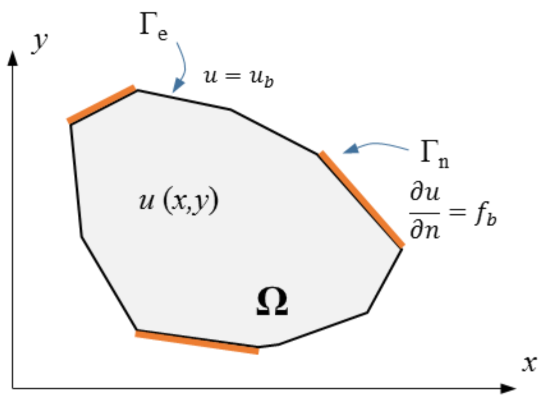


Fig. 1. Physical domain of Ω bounded by Γ

The PDE for a steady state heat conduction can be expressed as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \tag{1}$$

with the boundary conditions are

$$\left.\begin{aligned} u &= u_b && on \;\; \Gamma_e \\ \frac{\partial u}{\partial n} &= f_b && on \;\; \Gamma_n \end{aligned}\right\} \tag{2}$$

where u_b and f_b are the Dirichlet and Neumann boundary conditions, respectively. The weighted residual of the PDE and its boundary of Eq. (1) and (2) can be written as

$$I = -\int_{\Omega} w\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) d\,\Omega + \oint w\,\frac{\partial u}{\partial n} d\,\Gamma \tag{3}$$

where w is the weighted function formulated. By performing integration by part to Eq. (3), its weak formulation can be obtained as

$$I = -\int_{\Omega}\left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y}\right) d\,\Omega + \oint w\,\frac{\partial u}{\partial n} d\,\Gamma \tag{4}$$

Similar to the procedure of FEM given in [13], assume that the domain can be divided into a number of triangular elements as shown in Figure 2. Ref. 13 describes the procedure by using a linear basic function. In the present work, a quadratic basic function is used for each axis. Each triangular element has six nodes located on each vertex and mid-point of each side. The basic function of the element for this bi-quadratic element can be written as

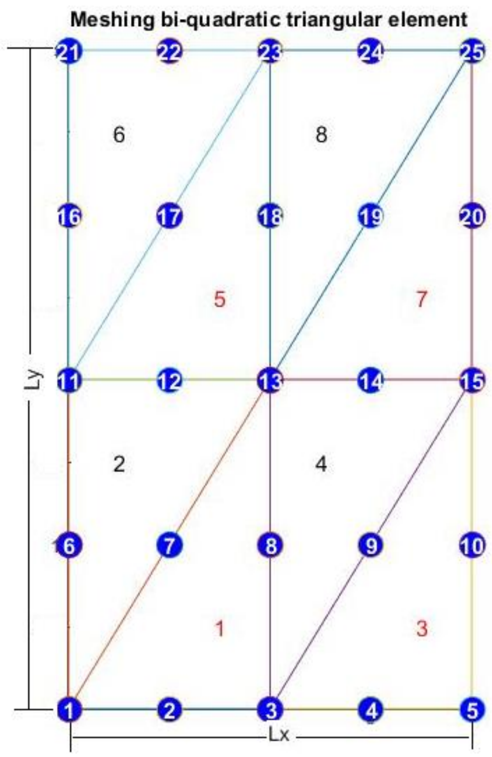


Fig. 2. Structured mesh using the six nodes, bi-quadratic triangular element model

$$u = a_i + a_i x + a_i y + a_i x^2 + a_i y^2 + a_i xy \tag{5}$$

Assume that the six node coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) and (x_6, y_6) and their six nodal variables are u_1, u_2, u_3, u_4, u_5 and u_6. The value of the variable u at arbitrary location (x, y) within the elemental triangular domain region is approximated by the basis function above which can be written in a matrix form as follows:

$$u = \begin{bmatrix} 1 & x & y & x^2 & y^2 & xy \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \tag{6}$$

where $a_i$is the constant to be figured out. The basis function of Eq. (6) should satisfy the nodal variables at the six nodal points. By substitution of the x and y values at each nodal point gives into Eq (6) gives

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^{\ 2} & y_1^{\ 2} & x_1 y_1 \\ 1 & x_2 & y_2 & x_2^{\ 2} & y_2^{\ 2} & x_2 y_2 \\ 1 & x_3 & y_3 & x_3^{\ 2} & y_3^{\ 2} & x_3 y_3 \\ 1 & x_4 & y_4 & x_4^{\ 2} & y_4^{\ 2} & x_4 y_4 \\ 1 & x_5 & y_5 & x_5^{\ 2} & y_5^{\ 2} & x_5 y_5 \\ 1 & x_6 & y_6 & x_6^{\ 2} & x_6^{\ 2} & x_6 y_6 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \tag{7}$$

where the shape function H can be constructed from Eq. (7) as follows

$$u = H_1\ (x,y)\ u_1 + H_2\ (x,y)\ u_2 + H_3\,(x,y)u_3 + H_4(x,y)u_4 + H_5\,(x,y)u_5 + H_6\,(x,y)u_6 \tag{8}$$

The stiffness matrix element can be obtained by the integration of the derivative of the shape functions as follows:

$$K_{i,j}^{\ e} = \int_{\Omega^e} \left(\frac{\partial H_i}{\partial x} \frac{\partial H_j}{\partial x} + \frac{\partial H_i}{\partial y} \frac{\partial H_j}{\partial y} \right) d\,\Omega \tag{9}$$

To solve the integration above most of the literatures suggest to use a Jacobian matrix so that it can handle arbitrary form of triangle shape. However, for a structured mesh where the triangular element shape can be of a straight triangle form as shown in Figure 3, a direct integration is possible. In the present work, the direct integration is conducted resulting to a simple stiffness matrix as given in Eq. (10).

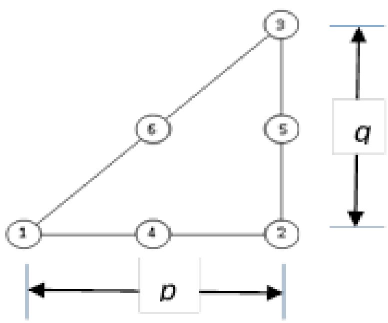


Fig. 3. Six nodes triangular element

$$
\left[K^e\right] = \begin{bmatrix} 3q^2 & q^2 & 0 & -4q^2 & 0 & 0 \\ & 3r^2 & p^2 & -4q^2 & -4p^2 & 0 \\ & & 3p^2 & 0 & -4p^2 & 0 \\ & & & 8r^2 & 0 & -8p^2 \\ & sym & & & 8r^2 & -8q^2 \\ & & & & & 8r^2 \end{bmatrix} \tag{10}
$$

where $r = \sqrt{p^2 + q^2}$

3. Results and Discussion

To show the features of the present GFEM, the method is used to perform numerical simulation of a rectangular plate having mixed asymmetric boundary conditions shown in Figure 4. The Dirichlet boundary conditions at the sides *AB* and *AD* of the plate are T = 0°C. The Neuman boundary condition at the side *BC* is dT/dx = 0. The Dirichlet boundary condition at the side *CD* is a sinusoidal function of $T = T_0 . \sin \frac{\pi x}{L_x}$.

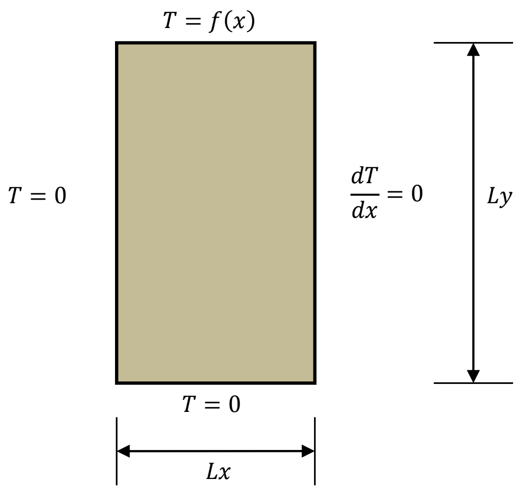


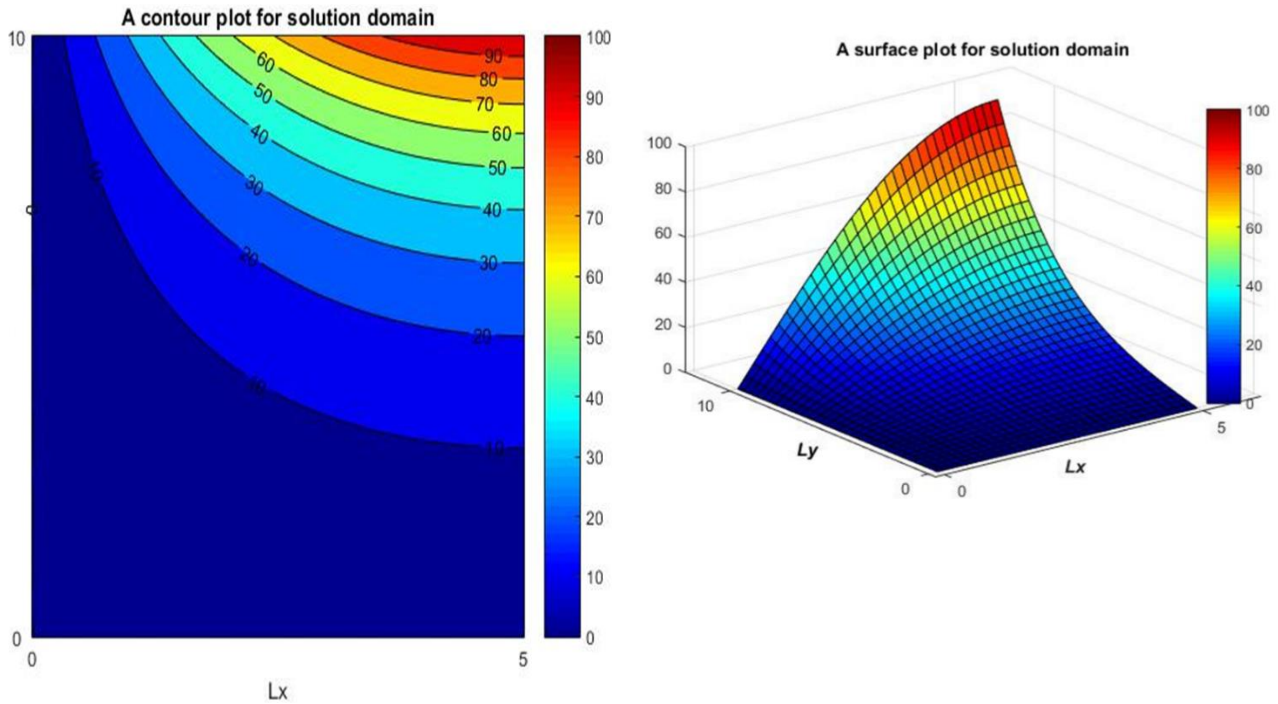
Fig. 4. Rectangular plate having mixed boundary conditions

This particular problem is introduced in [12] for validation of their work, where the dimension of the plate are fixed at L_x = 5 and L_y= 10 cm and T_0 = 100° C. For the purpose of validation of the present work, where the dimensions L_x and L_y may need to vary, the analytical solution to this problem is derived as shown in Appendix. The derivation shows that the exact temperature distribution on the plate is

$$
T = T_0 \sin(\alpha x) \frac{\sinh(\alpha y)}{\sinh(\alpha L_y)} \tag{11}
$$

where $\alpha = \frac{\pi}{L_x}$

The numerical simulations are conducted by varying the number of elements along x and y direction. The contour plot and surface plot presented in Figure 5a and 5b respectively shows the temperature distribution for $n_x = n_y = 32$. The temperature distribution behaves smoothly on domain as expected. The error contour plots for $n_x = n_y = 5$ and $n_x = n_y = 10$ are presented in Figure 6a and 6b respectively. The error here is defined as the temperature difference between the exact and numerical solutions for each point. Figure 6 shows that the order of error is drastically reduced from the maximum of 0.01 to 0.006 that demonstrate the convergence of the present method. Increasing the number of elements also reduces the error covering area.



(a) (b)

Fig. 5. Temperature distribution using QGFEM for $n_x = n_y = 32$

To check the convergence and its accuracy, the error can be presented as function of matrix norm and the results as presented in Table 1 and Figure 6. Adjerid *et al.,* [14] use the L^1 norm for their error estimation. Yi [15] use the L^∞ norm for the error estimate of the hp continuous Petrov Galerkin method. Wihler [16] use both L^1 and L^2 norms in his work on the continuous Galerkin FEM. Similar to Ref. [17], in the present work three matrix norms L^1 , L^2 and L^∞ are used to show the non-dimensional error of the present QGFEM where all norm are computed numerically subsequent to the mesh used. The three norm quantities are defined in Eq. (12).

$$H^1 = \frac{\sum_i^N |T_i - u_i|}{N}$$

$$L^2 = \sqrt{\frac{\sum_i^N (T_i - u_i)^2}{N}} \tag{12}$$

$$L^{\infty} = \max \sum_{i}^{N} |T_i - u_i|$$

where T is the exact temperature and u is the temperature calculated using the present QGFEM. The results presented in Table 1 are using non dimensional quantity where the temperature is normalized by T_o which is the maximum temperature at the boundary condition. Table 1 shows that increasing the number of elements or reducing the element size reduces significantly the error.

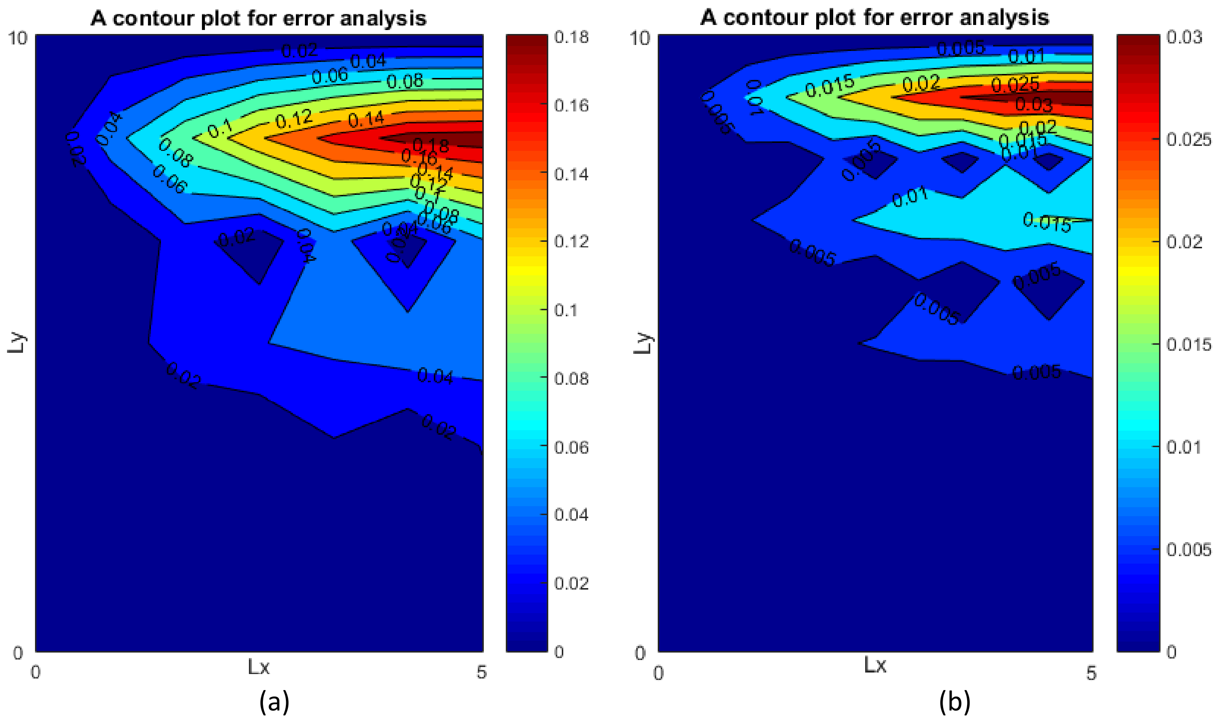


Fig. 6. The error of the temperature distribution using QGFEM for $n_x = n_y = 32$

Table 1
L^{∞}, L^2 and L^1 Error Analysis of the solution

Number of elements $n_x \times n_y$	Element size h	L^1	L^2	L^{∞}
8 × 8	0.8839	$6.9209\ 10^{-4}$	$1.3421\ 10^{-3}$	$5.9827\ 10^{-3}$
16 × 16	0.4419	$4.6060\ 10^{-5}$	$8.6499\ 10^{-5}$	$4.2211\ 10^{-4}$
24 ×24	0.2946	$9.2828\ 10^{-6}$	$1.7183\ 10^{-5}$	$8.6637\ 10^{-5}$
32 ×32	0.2210	$2.9786\ 10^{-6}$	$5.4389\ 10^{-6}$	$2.7937\ 10^{-5}$
40 ×40	0.1768	$1.2422\ 10^{-6}$	$2.2250\ 10^{-6}$	$1.1576\ 10^{-5}$
48 ×48	0.1473	$6.1697\ 10^{-7}$	$1.0769\ 10^{-6}$	$5.6273\ 10^{-6}$
56 ×56	0.1263	$3.5833\ 10^{-7}$	$5.9627\ 10^{-7}$	$3.0563\ 10^{-6}$
64 ×64	0.1105	$2.4253\ 10^{-7}$	$3.7828\ 10^{-7}$	$1.8008\ 10^{-6}$
72 ×72	0.0982	$1.8556\ 10^{-7}$	$2.7832\ 10^{-7}$	$1.1295\ 10^{-6}$

To assess the rate of convergence, the error norms are plotted as function of the element size as shown in Figure 6. By using a general regression to approximate the trend of the error norms as functions of the element size, the results show that the errors of the QGFEM can be generated as

$$e_1 = 0.0010h^{3.8097}$$

$$e_2 = 0.0021h^{3.9149} \tag{13}$$

$$e_\infty = 0.0100^{3.910}$$

where h is the average size of the triangular element calculated as

$$h = \sqrt{\frac{L_x \times L_y}{n_x \times n_y}} \tag{14}$$

The result shows that the present formulation convergence rate is nearly 0 $(h)^{3.8}$ which is a promising achievement to increase the accuracy of the simulation (Figure 7).

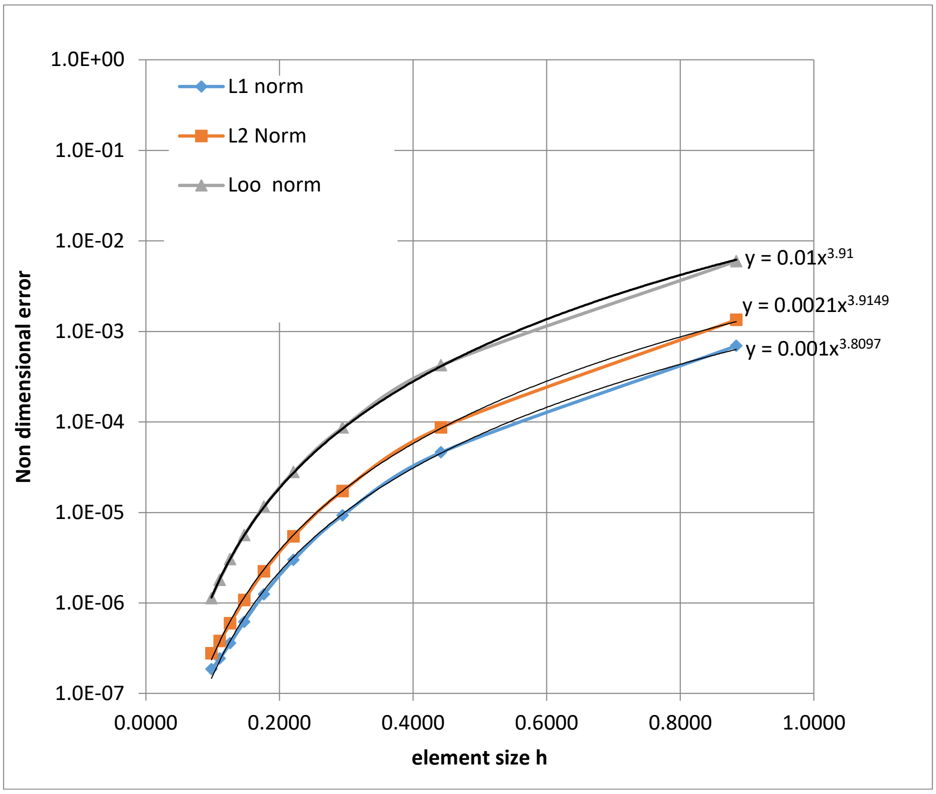


Fig. 7. Non dimensional error norm plot as function of element size

4. Conclusions

The present Galerkin's finite element method is developed based on a six-nodes, bi- quadratic triangular element model. The stiffness matrix of the element is formulated in a closed form solution such that it does not need additional procedure to evaluate its Jacobian matrix. The accuracy of the present model is demonstrated by comparison with analytical solution. The analytical solution of a heat conduction problem of a rectangular plate with an asymmetric, mixed boundary conditions is selected as a bench mark case. The error analysis and convergence study shows that the present GFEM convergence result is in the order of 0 $(h)^{3.8}$.

Acknowledgement
The support of International Islamic University Malaysia under the research grant RIGS17-033-0608 is gratefully acknowledged.

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