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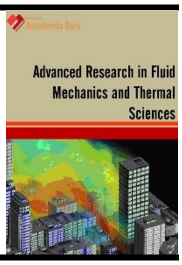

Analysis of Fluid Flow Between Two Rotating Disks

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ABSTRACT

The viscous incompressible fluid is considered between two disks which are spaced a distance $H\surd(1-\alpha t)$ and rotating with angular velocities. The governing Navier-Stokes equations reduced to a pair of nonlinear differential equations. We obtain the solution to these equations by computer extended perturbation series solution with special reference to normal forces and torques. The coefficient of the parameter $(-g'(0))$ decreasing in magnitude and alternating sign. By Domb-sykes plot, the singularity is identified. The series is recasted using Euler transformation. But the coefficients of $f'''(0)$ decreasing in magnitude having fixed sign pattern is recasted using reversion.

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1. Introduction

The steady motion of viscous fluid between two rotating disks with a fixed distance has many applications. The transformations reduces the Navier-Stokes equations to a set of nonlinear differential equations. The transformation was first described by T. V. Karman [1], followed by Batchelor [2]. The study of a three dimensional viscous flow problem which is also studied by Stewartson [3], Holodniok *et al.,* [4]. When the disks rotate with time dependent angular velocities ω_1 (t) and ω_2 (t) Karman transformation is applicable. The special cases arising out of these are studied by Ishizawa [5], Macadonald *et al.,* [6]. Walter G. Kelley, A. C. Peterson [7] and P. L. Sachdev [8] have discussed many analytical methods to solve the differential equations and also [10,11].

In the literature, the study of fluid flow between two disks is restricted for certain parameters like velocity, pressure gradient. But in the present study, flow between two disks is analysed for the characteristics like torque and load effects through semi-numerical method. This type of flow is having more applications in engineering.

Most numerical methods are comparatively tedious and difficult to implement due to nonlinear nature of the problem. For simple geometries, the semi numerical methods have advantages over pure numerical methods. These methods reveal analytical structure of the solution

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Van Dyke [9,12]. Bujurke, Pai [13-16] and his associates have successfully explored the power of these methods. Here we calculate a sufficient number of universal coefficients of low Reynolds perturbation series using computer. The convergence of such series is limited by a singularity we identify the nature and location of the singularity. Using Euler transformation, reversion etc., the region of validity is extended.

2. Formulation of the Problem

The axisymmetric incompressible flow between two parallel disks with rotation and spaced a distance d (t) apart where t denotes time as shown in the Figure 1 below. Taking cylindrical coordinates r, θ, z in the direction u, v and w respectively. Assuming

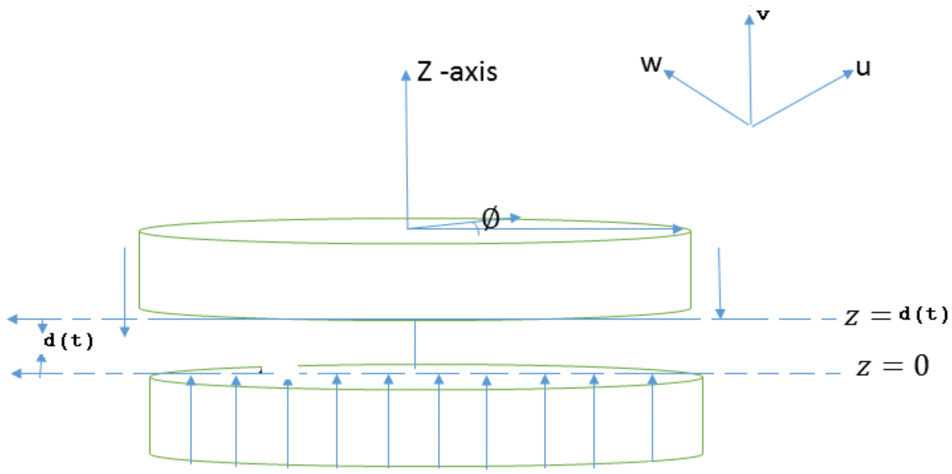

Fig. 1. Geometry of the problem

$W = -2F(z,t)$

$$u = r\frac{\partial F}{\partial z} \quad (1)$$

$\frac{\partial^2 p}{\partial r \partial z}=0, \ \frac{v}{r} = G(z,t).$

The nonlinear equations are obtained as follows

$$F_{zzt} - 2F_{zzzz} - 2GG_z = \gamma F_{zzzz} \quad (2)$$

and

$$G_t + 2FF_z - 2FG_z = \gamma G_{zz}$$

To find the similarity solution we change variables to

$$\zeta=t, \quad \eta = \frac{z}{d(t)} \quad , \ d(\zeta) = H\sqrt{(1-\alpha\zeta)} \quad (3)$$

The equations finally take the form,

$$\frac{f^{iv}}{R_e^S} = 3f'' + [\eta - 2f]f''' - 8\left(\frac{\omega_1}{\alpha}\right)^2 gg' \quad (4)$$

$$\frac{g''}{R_e^S} = 2g + \eta g' + 2gf' - 2fg'$$

$\omega_1(t)\&\omega_2(t)$ angular velocities of the disks. $\Omega_1(1-\alpha t)^{-1}$ & $\Omega_2\ (1-\alpha t)^{-1}$ denote the angular velocities of the disks. The parameter $\frac{\Omega_1}{\alpha}$ is equal to the ratio$(\frac{R_e^R}{2R_e^S})$, where $R_e^R = \Omega_1 H^2/\upsilon$ is a Reynolds number based on the speed of rotation of the disks $R_e^S = \alpha H^2/2\nu$ is a Reynolds number based on their speed of approach.
The boundary conditions are

$$u = 0, \qquad v = \frac{\omega_1 r}{1-\alpha t}, \qquad w = 0 \quad \text{on} \quad z = 0$$

$$u = 0, \qquad v = \frac{\omega_2 r}{1-\alpha t}, \qquad w = \frac{-\alpha H}{2\sqrt{(1-\alpha t)}} \quad \text{on} \quad z = H\sqrt{(1-\alpha\zeta)}.$$

Thus

$$f(0) = 0, \quad f'(0) = 0, \quad g(0) = 1$$
$$f(1) = \frac{1}{2}, \quad f'(1) = 0 \quad g(1) = s. \quad (5)$$

Finally the equations are approximated as

$$f^{iv} = R_e^s[3f'' + (\eta - 2f)\, f'' - 2(\frac{R_e^r}{R_e^s})\, gg'] \quad (6)$$

$$g'' = R_e^s\, [2g + \eta g' + 2gf' - 2fg']$$

subjected to the conditions (5).

3. Method of Solution

Assuming the solution of (6) in the form

$$\sum_{n=0}^{\infty}(\ R_e^s\ )^n f_n\ = f = f_0 + R_e^s f_1 + (R_e^s)^2 f_2 + \ldots$$

$$\sum_{n=0}^{\infty}(\ R_e^s\ )^n g_n = g = g_0 + R_e^s\ g_1 + (R_e^s)^2 g_2 + \ldots$$

with the boundary conditions

$$f_0\ (0) = f'_{\mathbf{0}}(0) = 0, \quad f_0(1) = \frac{1}{2}, \quad f'_0(1) = 0$$

$$f_n\ (0) = f'_n(0) = 0, \quad f_n(1) = \frac{1}{2}, \quad f'_n\ (1) = 0\ for\ n \geq 1$$

and

$$g_0(0) = 1, \qquad g_0(1) = \frac{\omega_2}{\omega_1} = s$$

$$g_n(0) = 1, \qquad g_n(1) = \frac{\omega_2}{\omega_1} = 0 \quad for \;\; n \geq 1.$$

On solving

$$f_0(\eta) = \frac{1}{60}(90\eta^2 - 2R_e^r s^2\eta^2 - 60\eta^3 + 3R_e^r s^2\eta^3 - R_e^r s^2\eta^5)$$

$$f_1(\eta) = \frac{1}{4536000}(307800\eta^2 + 74970R_e^r s^2\eta^2 + 332(R_e^r)^2 s^4\eta^2 - 1263600\eta^3 - 125040R_e^r s^2\eta^3 - 125040R_e^r s^2\eta^3 - 579(R_e^r)^2 s^4\eta^3 + 1701000\eta^4 - 37800R_e^r s^2\eta^5 - 907200\eta^5 + 128520R_e^r s^2\eta^5 + 192(R_e^r)^2 s^4\eta^5 + 226800\eta^6 - 16380R_e^r s^2\eta^6 + 252(R_e^r)^2 s^4\eta^6 - 64800\eta^7 - 25920R_e^r s^2\eta^7 - 162(R_e^r)^2 s^4\eta^7 + 1350R_e^r s^2\eta^8 - 30(R_e^r)^2 s^4\eta^8 + 300R_e^r s^2\eta^9 - 15(R_e^r)^2 s^4\eta^9 + 10(R_e^r)^2 s^4\eta^{11})$$

$$g_0(\eta) = s\eta$$

$$g_1(\eta) = \frac{1}{6300}(-3465s\eta - 8R_e^r s^3\eta + 3150s\eta^3 + 1575s\eta^4 - 35R_e^r s^3\eta^4 - 1260s\eta^5 + 63R_e^r s^3\eta^5 - 20R_e^r s^3\eta^7)$$

It is very much essential to get higher order approximations in the series if it has to reveal the true nature of the function represented by it. As we move to higher approximations the algebra becomes tedious and difficult to calculate the terms manually. So we use a systematic scheme to generate the terms of the order n = 25.

4. Analysis and Improvement of the series

The expression for torque and load in terms of series as follows

$$g'(0) = \dot{T}u = \sum_{n=0}^{\infty} c_n (R_e^s)^n \tag{7}$$

$$g'(1) = \dot{T}L = \sum_{n=0}^{\infty} d_n (R_e^s)^n \tag{8}$$

And

$$\dot{W} = -\frac{1}{R_e^s}[f'''(0) + 4(\frac{\omega}{\alpha})2R_e^s] \tag{9}$$

where

$$f'''(0) = \sum_{n=0}^{\infty} c'_n (R_e^r)^n \tag{10}$$

As the series (7, 8, 10) are slow converging it is essential to get higher approximations to analyze the problem. By using the Mathematica programing we generated the 25 approximations.

Coefficients of the series $-g'(0)$ (Table 1) are decreasing in magnitude and have alternate sign pattern. The nearest singularity, lying on the negative axis has no direct physical significance. In this case, the simplest device to use is an Euler transformation based on estimate of ϵ_0 the radius of convergence of the series (7). With this transformation, the singularity is mapped to infinity. The transformation envisages using the new variable ϵ^* such that

$$\epsilon^* = \frac{R_e^s}{R_e^s + \epsilon_0} \quad \text{or} \quad R_e^s = \frac{\epsilon_0 \epsilon^*}{1-\epsilon^*}$$

Table 1
Coefficient of the series $-g'(0)$ for $s = 0$ and $R_e^r = 1$

n	c_n	n	c_n
0	1.00000000	13	3.26205231E-006
1	0.95428571	14	-1.22458693E-006
2	-0.23976509	15	4.62690852E-007
3	0.08184054	16	-1.75915635E-007
4	-0.02894860	17	6.72833224E-008
5	0.01032749	18	-2.58793563E-0.08
6	-0.00370138	19	1.00064754E-008
7	0.00133243	20	-3.88795961E-009
8	-0.00048199	21	1.51742068E-009
9	0.00017523	22	-5.94660546E-010
10	- 0.00006413	23	2.33913170E-010
11	0.00002361	24	-9.23245072E-011
12	8.74670646E-006	25	3.65529660E-011

The series (7) takes into a new form

$$-g'(0) = \sum_{n=0}^{\infty} b_n \epsilon^* \tag{11}$$

where the coefficients $b_0 = 1{:}0000$

$$b_n = \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!(j-1)!} c_j \epsilon_0^j$$

The first few coefficient can be written as

$$-g'(0) = c_0 + (c_1\epsilon_0 + c_2\epsilon_0)\epsilon^{*^2} + (c_1\epsilon_0 + 2c_2\epsilon_0{}^2 + c_3\epsilon_0{}^3)\epsilon^{*^3} + \ldots$$

The new series (11) can be used to approximate the solution up to R_e^s = 1000. The similar analysis is carried for ($R_e^r = 10$,$R_e^r = 20$) for ($s = 0, s = -1, and\ s = 1$). Also for $f'''(0)$ when $s = -1, R_e^s = 0.1, 0.5$ and $s = 0, R_e^s = 0.1$.

The coefficients of the series for $f'''(0)$ (Table 2) are decreasing in magnitude and have fixed negative sign pattern. Singularity lies on the positive axis, it indicates that the mathematical model has broken down in one of the several ways. It is possible to increase the numerical accuracy of the series. The artificial restriction it impose on convergence can be eliminated by reverting the role of independent and dependent variable.

Table 2
Coefficient of the series – $g'(1)$ *and* $f'''(0)$ *for* $s = 0$ *and* $R_e^s = 10$

n	c'_n	n	c'_n
0	-21.86179238	13	-0.02899299
1	-2.42477372E+014	14	-0.00025262
2	-2.81108729E+014	15	-1.85210977E-006
3	-7.70377264E+13	16	-1.14207312E-008
4	-9.58678075E+012	17	-5.90665318E-011
5	-7.09112596E+011	18	-2.54871912E-013
6	-3.56270780E+010	19	-9.10031782E-016
7	-1.31162400E+009	20	-2.65649569E-018
8	-3.70968932E+007	21	-6.23092560E-021
9	-831680.79654572	22	-1.14507453E-023
10	-15100.72169882	23	-1.58701764E-026
11	-225.40954460	24	-1.55890818E-029
12	-2.79510884	25	-9.66851857E-033

The analysis gives

$$f'''(0) = -c'_0 + c'_1 R_e^r + c'_2 (R_e^r)^2 + c'_3 (R_e^r)^3 + \ldots$$

$$Y' = c'_1 R_e^r + c'_2 (R_e^r)^2 + c'_3 (R_e^r)^3 + \ldots$$

where $Y' = f'''(0) + c'_0$

$$R_e^r = b_1 Y' + b_2 Y'^2 + b_3 Y'^{\mathbf{3}} + \ldots .$$

Now

$$Y' = c'_1 b_1 Y' + (c'_1 b_2 + c'_2 b_1{}^2)\, Y'^2 + (c'_1 b_3 + 2c'_2 b_1 b_2)\, Y'^{\mathbf{3}}$$

equating the coefficients gives

$$b_1 = c'^{-1}_1$$

$$b_2 = -c'_2 c'^{-3}_3$$

$$b_3 = c'^{-5}_1 (2c'^2_2 - c'_3 c'_1)$$

The coefficients of the series for $f'''(0)$, (Table 3) are random in sign, to improve the accuracy of the results we use the Pade-approximants. The basic idea of Pade-approximants is to replace power series $\sum_{n=0}^{\infty} c'_n (R_e^r)^n$ by a sequence of rational function of the form

$$P_M^N(R_e^r) = \frac{\sum_{n=0}^{N} A_n (R_n^R)}{\sum_{n=0}^{M} B_n (R_e^r)}$$

where we choose B0 = 1 without loss of generality. We determine the remaining (M + N + 1) coefficient $A_0, A_1, A_2 \ldots A_N$. $B_1, B_2, B_3, \ldots B_M$ so that the first (M + N + 1) terms in the Taylors series expansion of $P_M^N(R_e^r)$ match with first (M + N + 1) terms of the power series $\sum_{n=0}^{\infty} c'_n (R_e^r)^n$. The resulting rational function $P_M^N(R_e^r)$ is called a Pade-approximants.

Table 3
Coefficient of the series $f'''(0)$ $for\ s = -1\ and\ R_e^s = 1$

n	c_n	n	c_n
0	-7.65987047	13	1.96026658E-042
1	-0.81698571	14	-8.32842697E-046
2	- 0.00006591	15	3.47904299E-049
3	2.21320849E-008	16	-1.73898804E-052
4	-7.76199022E-012	17	8.53989228E-057
5	2.83745426E-015	18	-1.51308831E-058
6	-1.07203465E-018	19	-2.02352545E-061
7	4.15679564E-022	20	-3.07838749E-064
8	-1.64552184E-025	21	-3.36068395E-067
9	6.62461192E-029	22	-2.95611813E-070
10	-2.70436275E-032	23	-1.92358220E-073
11	1.11698951E-035	24	-8.99656513E-077
12	-4.65976795E-046	25	-2.66561525E-080

5. Results and Discussions

The flow of a viscous incompressible fluid between two rotating disks is governed by a couple of nonlinear ordinary differential Eq. (4) together with boundary conditions (5). The proposed perturbation series scheme enables us to obtain the large number of coefficients. A carefully written Mathematica code makes it possible to perform the complex algebra. The coefficients of the series – g' (0) decreases in magnitudes and alternate in sign. This indicates the presence of a singularity. A Domb-Sykes plot (Figure 1) provides nearest singularity.

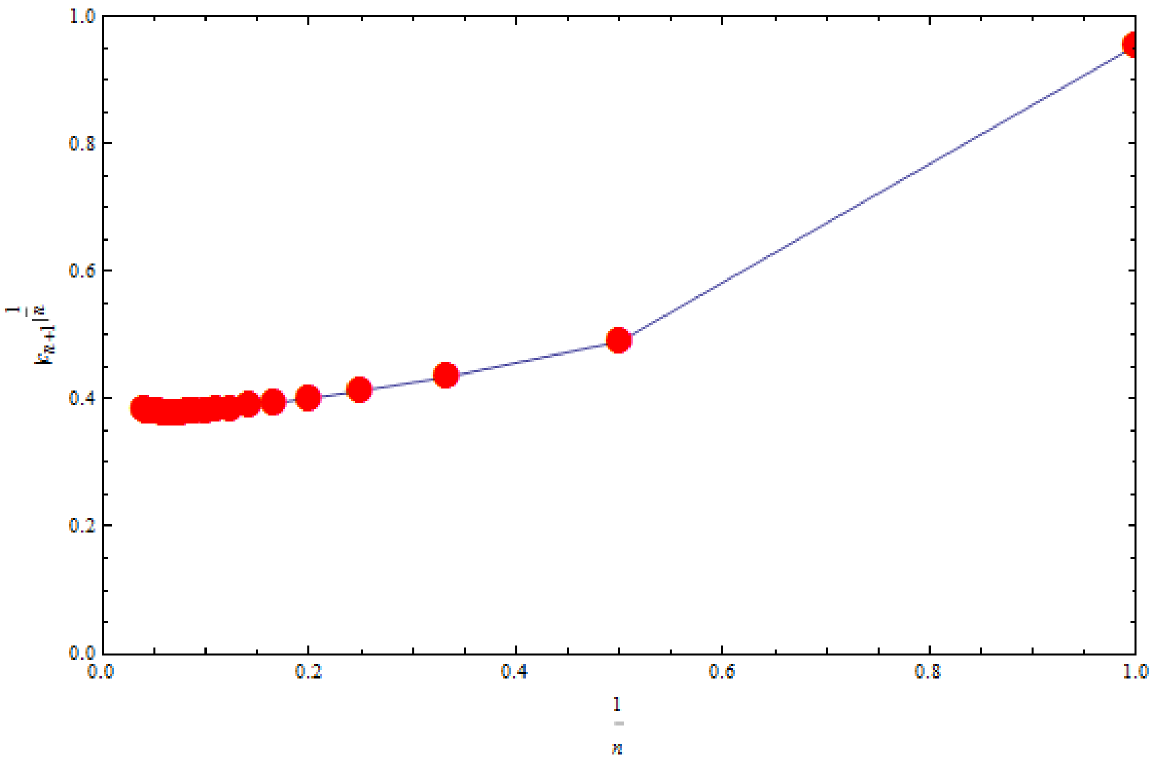

Fig. 2. Domb-Sykes plot for – g' (0) when s = 0, R_e^r = 1 ($\epsilon_0 = 2.5$)

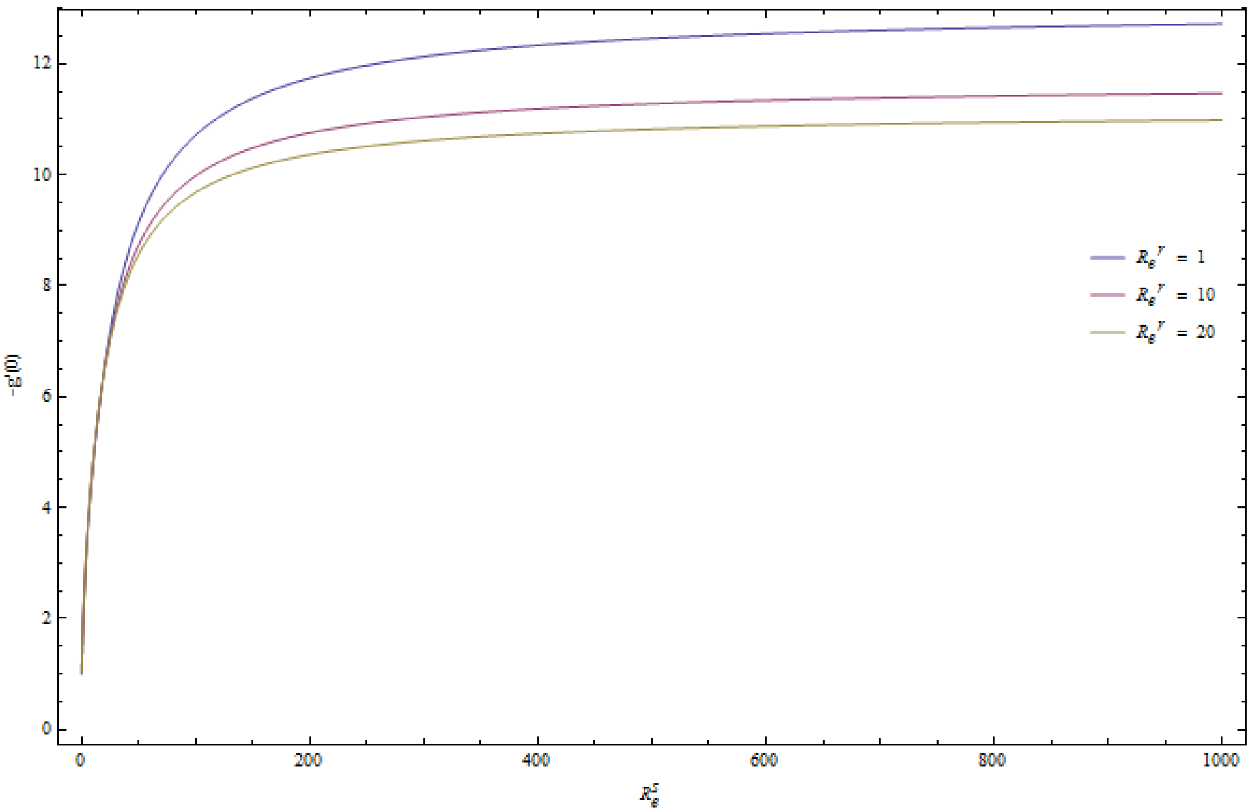

Fig. 3. Variation of torque at lower disk $-g'(0)$ for different values of R_e^s when s = 0

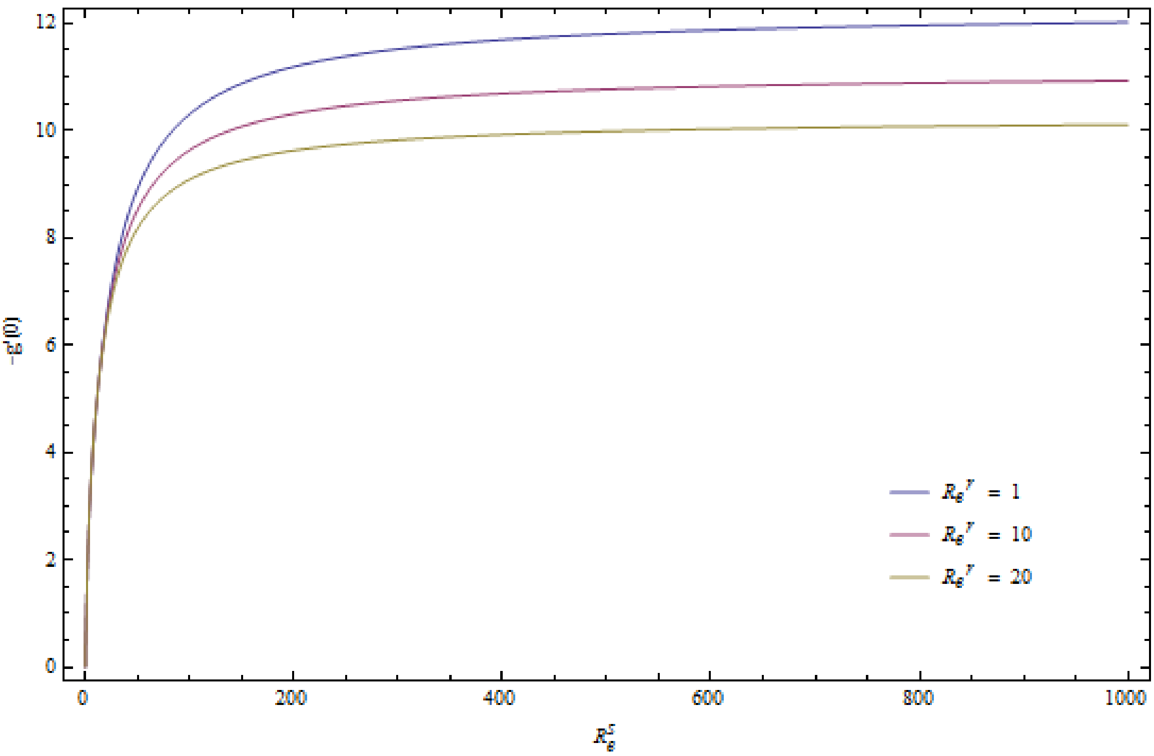

Fig. 4. Variation of torque at lower disk $-g'(0)$ for different values of R_e^s when s = 1

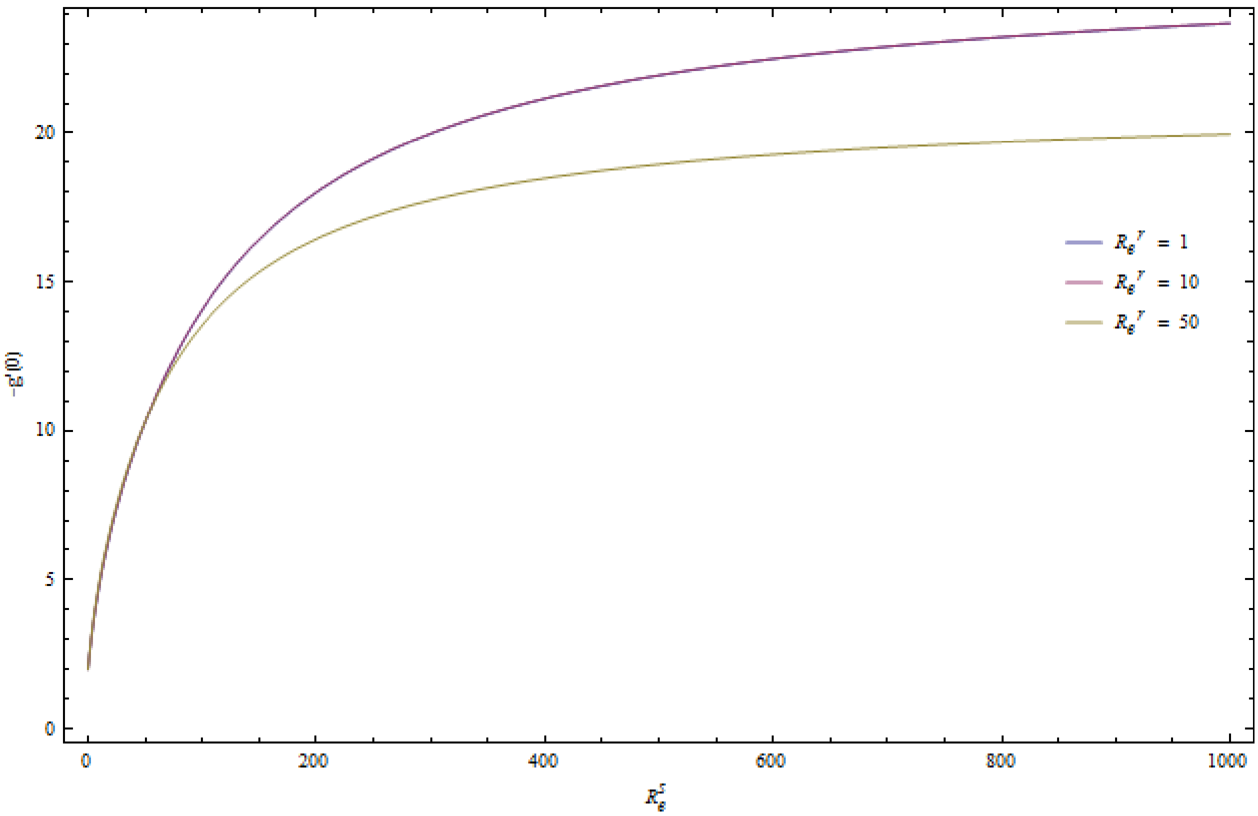

Fig. 5. Variation of torque at lower disk $-g'(0)$ for different values of R_e^s when s = -1

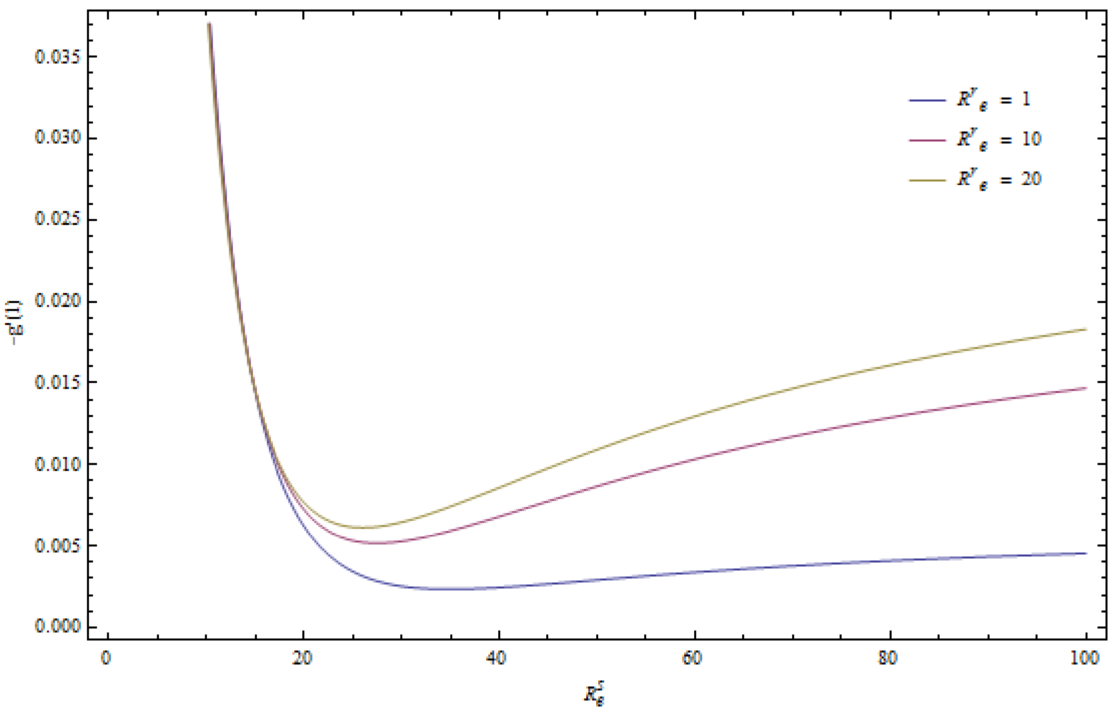

Fig. 6. Variation of torque at upper disk $-g'(1)$ for different values of R_e^s when s = 0

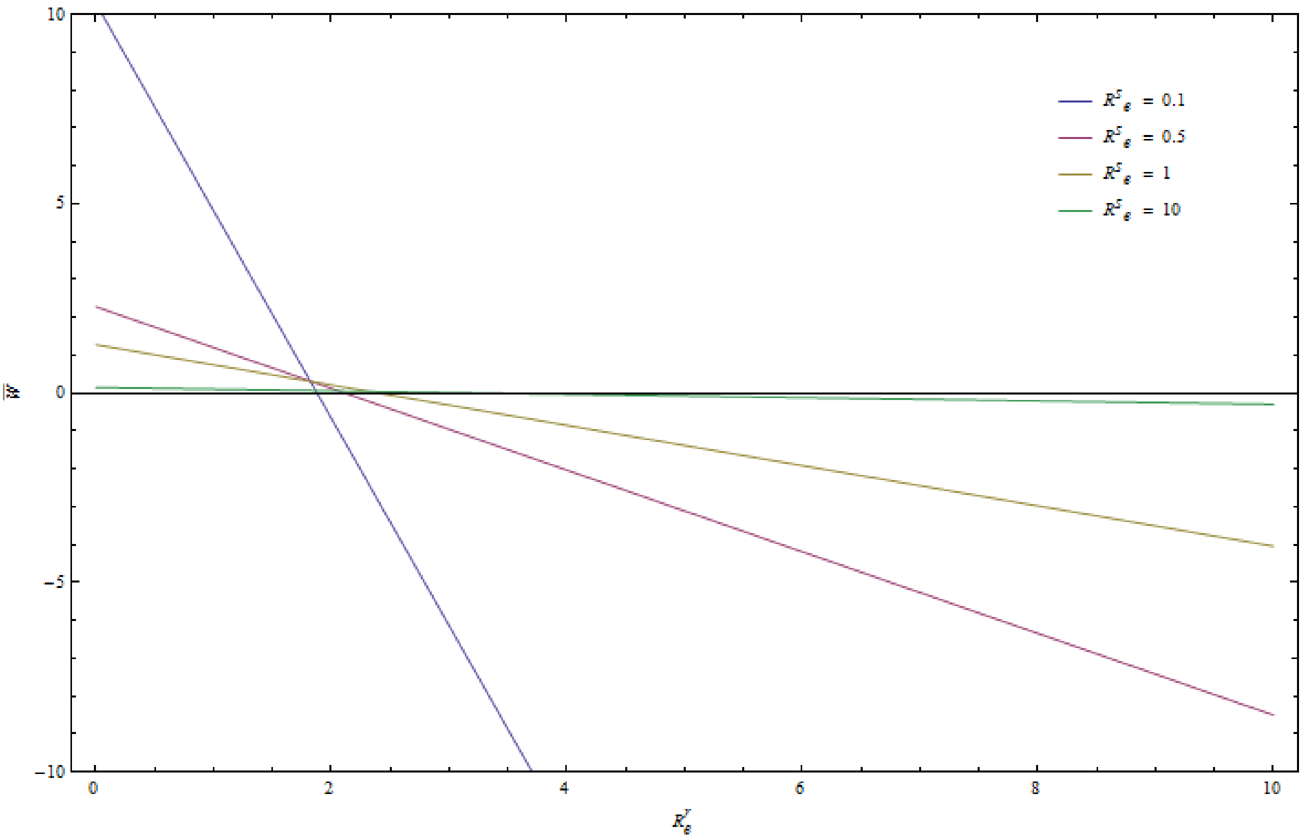

Fig. 7. Variation of load Ẇ for different values of R_e^r when s = 0

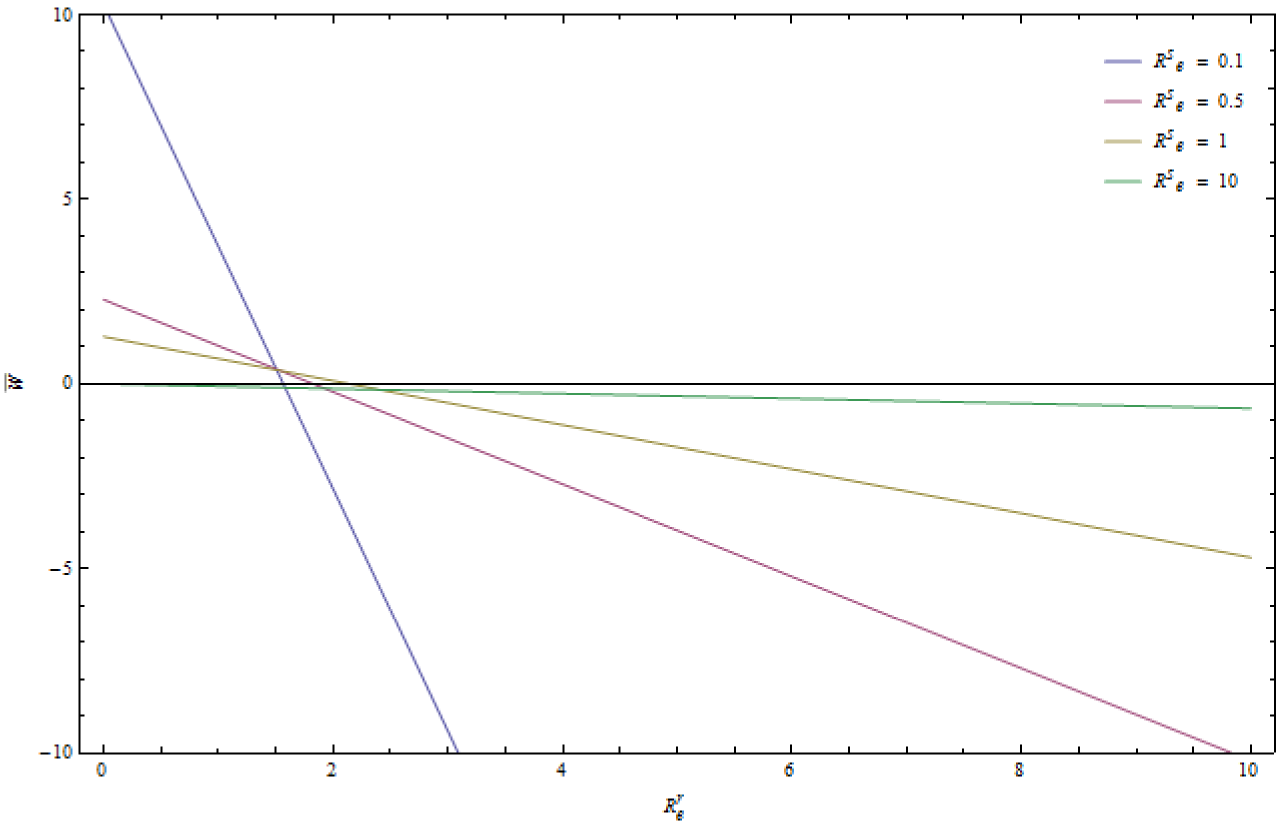

Fig. 8. Variation of load Ẇ for different values of R_e^r when s = 1

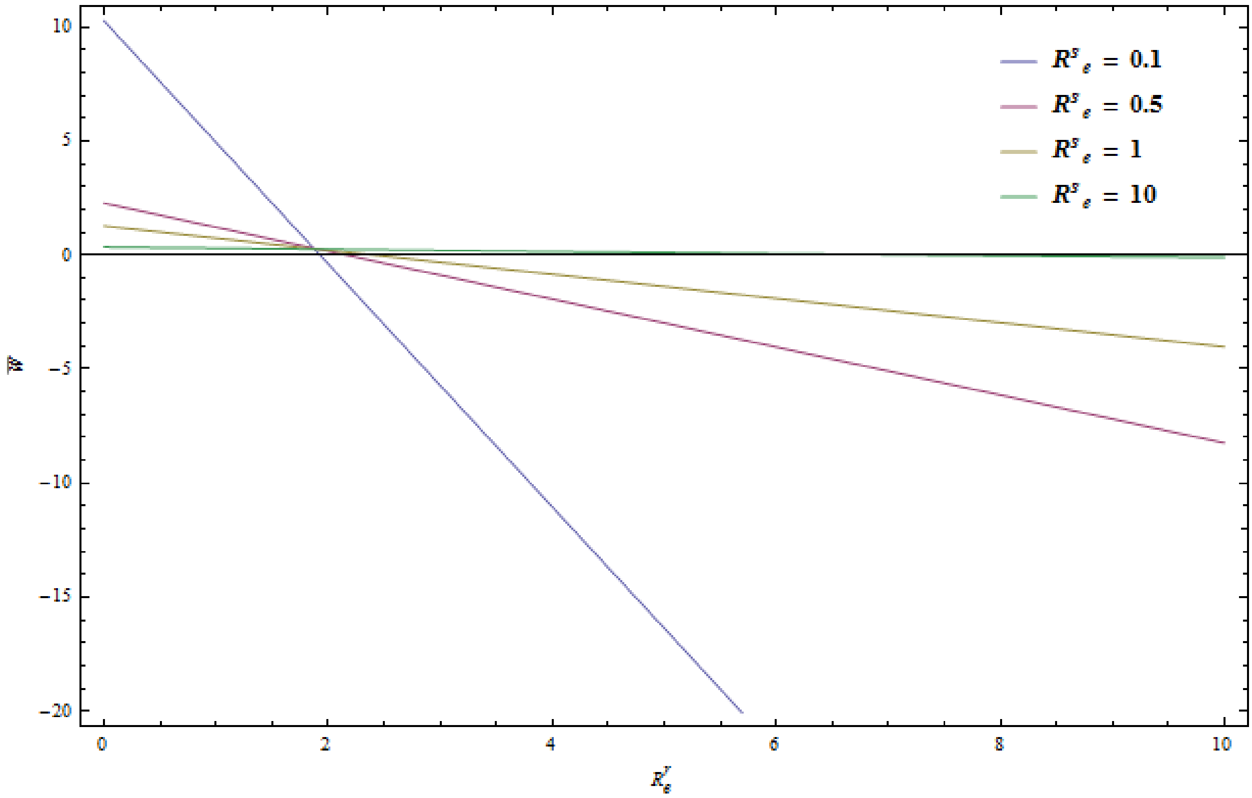

Fig. 9. Variation of load Ẇ for different values of R_e^r when s = -1

This singularity has no physical significance. The coefficients of the series f‴ (0), (Table2) decrease in magnitude has fixed sign pattern. Applying the reversion for recasting the series and further Pade-approximants are used to sum the series. Our results normal force and torque (Figure 2 to 9) are good in agreements with pure numerical results obtained by Macdonold [6]. Once the universal coefficients of the series are generated the rest of the analysis can be done at a single stretch, taking hardly any computer time and storage, while other numerical methods require huge storage and long computer time.

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