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Numerical Method for Solution of Fuzzy Nonlinear Equations

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ABSTRACT

Most studies on numerical techniques for solving fuzzy nonlinear equations focus on problems with non-singular Jacobian in the neighbourhood of the solution, particularly, the Newton's type methods. However, computing the Jacobian at every iteration is time consuming. In this paper, a derivative free numerical technique known as improved False Position algorithm was employed for the solution of fuzzy nonlinear equations. This is one of the simplest and reliable bracketing method for the finding the root of nonlinear equations. This method tends to overcome the computational cost of Jacobian evaluation at every iteration. The fuzzy quantities are transformed into parametric form. Numerical examples are provided to illustrate the efficiency of the method.

1. Introduction

Over the past decades, nonlinear systems have played important role in solving problems arising from numerous areas of application such as engineering, medicine, social sciences. However, the parameters of these systems of nonlinear equations are usually represented by fuzzy numbers rather than crisp numbers. Thus, the outcome depends on the roots of fuzzy equation [1]. Some standard analytical techniques such [4, 5], and [10] are not suitable for solving systems of nonlinear equations of the form

$$ay^3 + by^2 + cy - d = e \tag{1}$$

$$ae^y + b = d \tag{2}$$

where y, a, b, c, d and e are fuzzy numbers. Therefore, we need to develop numerical methods for finding the roots of such equations. [1] transformed the fuzzy quantities into its parametric form and employed Newton's approach to solve the equivalent system of fuzzy nonlinear equation. Newton method has a rapid rate of convergence when starting with an initial that is chosen close to the solution point. The disadvantage of this method is computing and inverting of the Jacobian matrix in

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every iteration. [2] applied the Quasi Newton's technique to compute the root of fuzzy nonlinear equations. This method also requires the storage of approximate of Jacobian matrix at every iteration. Also [11] proposed the Chord Newton's method to obtain the root of dual fuzzy nonlinear equations. However, this method also need the computation of Jacobian either at first iteration, or after few iterations. To overcome some of these drawbacks, we propose numerical solution of fuzzy nonlinear equation by improve False Position method. This method is one of the simplest and reliable bracketing method for the finding the root of nonlinear equations. It simplicity is due to the fact it does not require computation of any derivative.

This paper is structured as follows. In section 2, we present some of the preliminary result on arithmetic fuzzy operations. The improve False Position method for solving nonlinear equation is presented in section 3. In section 4, we introduce an algebraic fuzzy equation and provided the numerical solution of some examples to illustrate the efficiency of the propose method. The conclusion of the work is presented in section 5.

2. Preliminaries

In this section, we recall useful definitions of fuzzy numbers.

Definition 2.1 [7, 12].
A fuzzy number is a set like $u: R \to I = [0,1]$ which satisfy the following
1. u is upper semi-continuous
2. $u(x) = 0$ outside some interval $[c, d]$
3. there are real numbers a, b such that $c \le a \le b \le d$ and
   3.1 $u(x)$ is monotonic increasing on $[c, a]$
   3.2 $u(x)$ is monotonic decreasing on $[b, d]$
   3.3 $u(x) = 1, a \le x \le b$

The set of all fuzzy numbers is denoted by E. An equivalent parametric is also given in [8].

Definition 2.2 [7, 12]
Fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$of function $\underline{u}(\propto)$, $\overline{u}(\propto)$, $0 \le\propto\le 1$ which satisfies the following requirement;
(1) $\underline{u}(\propto)$ is a bounded monotonic increasing left continuous function,
(2) $\overline{u}(\propto)$ is a bounded monotonic decreasing left continuous function,
(3) $\underline{u}(\propto) \le \overline{u}(\propto), 0 \le\propto\le 1$.

Definition 2.3 [7]
A popular fuzzy number is the Triangular fuzzy number $u = (a, b, c)$ with membership function

$$u(x) = \begin{cases} \frac{(x-a)}{(c-a)}, & a \le x \le c, \\ \frac{(x-b)}{(c-b)}, & c \le x \le b, \end{cases} \tag{3}$$

where $c \ne a, c \ne b$. Its parametric form is

$$\underline{u}(\propto) = a + (c - a) \propto, \tag{4}$$

$$\overline{u}(\propto) = b + (c - b) \propto. \tag{5}$$

Let $TF(\mathbb{R})$ be the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows [12].

For arbitrary $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v})$, and $k > 0$, the addition $(u + v)$ and multiplication by scalar k are defined as

$$(\underline{u+v})(\propto) = \underline{u}(\propto) + \underline{v}(\propto)\,, \tag{6}$$

$$(\overline{u+v})(\propto) = \overline{u}(\propto) + \overline{v}(\propto), \tag{7}$$

$$(\underline{ku})(\propto) = k\underline{u}(\propto),\ (\overline{ku})(\propto) = k\overline{u}(\propto). \tag{8}$$

Definition 2.4 [3]. $\tilde{P}_n(x)$ is a fuzzy polynomial of degree at most n, if there are some fuzzy numbers $\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_n$ such that

$$\tilde{P}_n(x) = \sum_{j=0}^{n} \tilde{a}_j x^j \tag{9}$$

3. An Improved False Position Method

The False Position method also known as Regula Falsi method [13] is one of the earliest bracketing method for obtaining the roots of nonlinear equations. This method tends to improve the poor rate of convergence of the bisection method [9] as well as its poor adaptability to solve problem with higher dimensions. Though, False Position approach is one of the best methods, and would often be the best choice when solving nonlinear systems, that is, when Newton's method is being avoided due to the computational cost of evaluation of its derivative or when Newton's and other iterative methods have failed to converge. Numerous modifications have been proposed to improve this method. Given the False Position method

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)} \tag{10}$$

where x is known as the endpoint of $[a, b]$ and $f(a), f(b)$ have opposite signs. To obtain subsequent interval which bracket the root, we consider the following conditions. If,

$$\begin{aligned} &f(a) * f(b) < 0, \text{then } b = x \\ &f(a) * f(b) > 0, \text{then } a = x \\ &f(a) * f(b) = 0 \text{ then } x \text{ is the root.} \end{aligned} \tag{11}$$

The failure mode of this method can easily be detected as same end points can be retained twice or more in a row. To improve this method, [6] propose an improved version of the False Position method define as

$$x_n^{IFPM} = \frac{a_n f(b_n) - \frac{1}{2} b_n f(a_n)}{f(b_n) - \frac{1}{2} f(a_n)} \tag{12}$$

where IFPM stands for improve False Position method. This improved False Position method leads to a considerable enhancement in speed of convergence without the guarantee of convergence being lost.

Theorem 1 [9]. Let $x \in [a,b]$and also, suppose there exist a number, say $r \in [a,b]$ that satisfy $f(r) = 0$. If $f(a)f(b) < 0$, and

$$x_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)} \tag{13}$$

represents the sequence of points generated by False Position procedure, then the sequence $\{x_n\}$ converges to the zero $x = r$. That is $\lim_{x \to \infty} x_n = r$.

4. Fuzzy Equation

In this section, we give the definition of fuzzy equation and applied the improve False Position algorithm to solve the problem.
Definition 4.1 [3]. Fuzzy equation can be defined as follows

$$\tilde{P}_n(x) = \tilde{a} \tag{14}$$

where $\tilde{a}$ have the same LR as $\tilde{a}_j$ in $\tilde{P}_n(x)$.
Suppose we denote $\tilde{a}_j = (\underline{a}_j, \overline{a}_j)$ and $\tilde{a} = (\underline{a}, \overline{a})$ where

$$\underline{a}_j = \sum_{i=0}^{n} b_{ij}\, r^i,\ \overline{a}_j = \sum_{i=0}^{n} c_{ij}\, r^i, \tag{15}$$

$$\underline{a} = \sum_{i=0}^{n} g_i\, r^i,\ \overline{a} = \sum_{i=0}^{n} h_i\, r^i \tag{16}$$

and suppose x is a positive number, then we will have

$$\left(\sum_{j=0}^{n} \underline{a}_j\, x^j, \sum_{j=0}^{n} \overline{a}_j\, x^j\right) = (\underline{a}, \overline{a}) \tag{17}$$

which implies

$$\underline{a} = \sum_{j=0}^{n} \underline{a}_j\, x^j,\ \overline{a} = \sum_{j=0}^{n} \overline{a}_j\, x^j \tag{18}$$

Substituting (16) in (18), we have

$$g_i = \sum_{j=0}^{n} b_{ij}\, x^j, \quad h_i = \sum_{j=0}^{n} c_{ij}\, x^j \tag{19}$$

$i = o, 1, 2, \ldots, n$. Hence, we apply the improved False Position method to solve the above equation.

5. Numerical Illustrations

In this section, we perform numerical comparison of performance of improve False Position method (IFPM) with that of robust backtracking methods. The algorithms chosen for comparison are the False Position method (FPM), and the well-known Bisection algorithm. In an attempt to carry out the comparison over as representative of a problem as possible, an equation was constructed which possess characteristics commonly encountered in root-finding problems. The computation was performed using MATLAB (2013a) version on a corei5 MacBook Pro and terminated when when $\|f(x)\| \leq 10^{-5}$.

Example 1. Consider the following fuzzy equation

$$(r, 7-6r)x^2 + (r-4, 2-5r)x + (r+1, -3+5r) = (0.5r-0.5, 0.5-0.5r) \tag{20}$$

Let $a_0 = 0.9$, $b_0 = 1.05$. The result is presented in Table 1.

Example 2. Consider the following fuzzy equation

$$(r, 2-r)x^2 + (-8r+4, 2-6r)x + (-12r-9, -4-17r) = (5r-5, 5-5r) \tag{21}$$

Let $a_0 = 6.5$, $b_0 = 7.05$. The result is presented in Table 2.

False Position method and Bisection method are classified among the best bracketing methods for obtaining the roots of equations. However, from Table 1 and 2 above, the Improved False Position method perform better than these classical methods as it was able to obtain the root of the problems after 4 iterations, the classical False Positon method obtained the approximate root of the equations after 4 iterations, and Bisection method obtained the approximate root after 12 iterations and 17 iterations respectively. The obtained $\hat{x}$ for problem 1 is the root of $x^2 - 3x + 2 = 0$. Hence, $x^* = 1$ is the root of the fuzzy equation. Also, the obtained $\hat{x}$ for problem 2 is the root of $x^2 - 4x - 21 = 0$. Therefore, $x^* = 7$ is the root of the fuzzy equation.

Table 1
Solution of fuzzy nonlinear equation by IFPM, FPM, and Bisection method

Iteration	False Position Method		Bisection Method	
no (n)	x_n	$f(x_n)$	x_n	$f(x_n)$
0	0.980488	0.019893	0.975000	0.025625
1	1.001006	-0.001005	1.012500	-0.012344
2	1.000019	-0.000019	0.993750	0.006289
3	1.000000	-0.000000	1.003125	-0.003115
4			0.998438	0.001565
5			1.000781	-0.000781
6			0.999609	0.000391
7			1.000195	-0.000195
8			0.999902	0.000098
9			1.000049	-0.000049
10			0.999976	0.000024
11			1.000012	-0.000012
12			0.999994	0.000006

6. Conclusion

Recently, there have been wide range of computationally efficient numerical algorithms available for the solution of fuzzy nonlinear equations. However, for difficult problems in which information on the location of the root is poor, such methods often converge slowly or sometimes fail to converge, and this has led to a search for methods which are relatively insensitive to the choice of initial values. In this paper, an improved False Position approach was employ to obtain the root of fuzzy nonlinear equations. Numerical examples illustrate the efficiency of the proposed method.

Table 2
Solution of fuzzy nonlinear equation by IFPM, FPM, and Bisection method

Iteration no (n)	False Position Method x_n	$f(x_n)$	Bisection Method x_n	$f(x_n)$
0	6.953953	-0.458349	6.775000	-2.199375
1	6.999770	-0.002301	6.912500	-0.867344
2	6.999999	-0.000011	6.981250	-0.187148
3	7.000000	-0.000000	7.015625	0.156494
4			6.998437	-0.015623
5			7.007031	0.070362
6			7.002734	0.027351
7			7.000586	0.005860
8			6.999512	-0.004883
9			7.000049	0.000488
10			6.999780	-0.002197
11			6.999915	-0.000854
12			6.999982	-0.000183
13			7.000015	0.000153
14			6.999998	-0.000015
15			7.000007	0.000069
16			7.000003	0.000027
17			7.000001	0.000006

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