Richards' Equation: Transition Between Constitutive Equations and the Mechanics of Water Flow in Unsaturated Soil

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ABSTRACT

Water flow in soil is normally governed by Richards' partial differential equation. Similarly, the equation can be used to investigate water infiltration in the soil. It is well understood that when water hits the surface of the topsoil, it infiltrates into the soil. However, the mechanics behind the infiltration of water is unknown to many researchers. This study aims to reveal the mechanics behind the well-known Richards' equation that has been used enormously in governing water flow in unsaturated soil since 1931. A classic case study of Haverkamp's water infiltration into Yolo Light Clay was used in the study. The Richards' equation has been discretized using finite difference method and the algebraic solution has been coded into Simply Fortran 2008. The partial differential equation of Richards is supported by two constitutive functions. The functions from Haverkamp and van Genuchten were compared. It is often for researcher to change between the hydraulic functions, which have limitation on the simulation outcome. The solution to overcome the limitation by changing the constitutive function variables was provided.

Keywords:
water flow in soil; unsaturated soil;
water flux; water adsorption

1. Introduction

Richards' equation consists of a governing equation that is used to describe water flow in subsurface porous media such as soil [1]. It is continuously subjected to numerical investigation [2-9].The modeling of water distribution using the equation has important application in climate science, agriculture and also ecosystem management [10]. Among many applications, Zeide [11] has reported the use of Richards' equation in the tree growth modeling prediction. Soil scientist, agronomist, and irrigation engineer have found the equation useful to relate to plant water intake, and fertilizers and pesticides advection and dispersion in the soils [12].

The governing equation is represented by a partial differential equation. The equation is nonlinear that the soil water content (θ_L) has to be determined is influenced by its hydraulic conductivity (K) and matric suction (ψ_m), which both of them are depending on the soil water

content [13]. Adding to the complexity, Richards' equation governs the interface (unsaturated soil) is constantly subjected to the changing upper layer atmosphere and the changing lower layer of saturated soil that is governed Darcy's law. Richards' equation consists of two parts; the water flux expression that is similar to Darcy law, and the continuity equation is used to monitor the mass balance of water flux in and out of a control volume. The equation to describe the unsaturated flow can be written in three forms, as follows:

Head-based, $C(\psi_m)\dfrac{\partial \psi_m}{\partial t} = \dfrac{\partial}{\partial z}\left(K\dfrac{\partial \psi_m}{\partial z}\right) - \dfrac{\partial K}{\partial z}$ (1)

Mixed based, $\dfrac{\partial \theta_L}{\partial t} = \dfrac{\partial}{\partial z}\left(K\dfrac{\partial \psi_m}{\partial z}\right) - \dfrac{\partial K}{\partial z}$ (2)

Saturation-based, $\dfrac{\partial \theta_L}{\partial t} = \dfrac{\partial}{\partial z}\left(D(\theta_L)\dfrac{\partial \theta_L}{\partial z}\right) - \dfrac{\partial K}{\partial z}$ (3)

where detailed description on Eq. (1) was described in Rathfelder and Abriola [14]. Similarly, Kavetski *et al.,* [15] and Zeng and Decker [16] have provided a detail explanation for mixed- and saturation-based of Richards' equation. To implement the equation, the soil water retention relation (θ_L as a function of ψ_m) and the unsaturated hydraulic conductivity relation (K as a function of ψ_m) must be known from measurement [17].

Water infiltration is one of the common application for Richards' equation, and the water infiltrated medium is normally assumed homogeneous and isotropic [18]. In reality, the water flux regime could be affected depending on the locations in the soil that could be resulted from different transport coefficients such as the soil water retention curve and hydraulic conductivity relations, and also, a spatial variation of different saturated volumetric water contents [19]. Adding to this complexity, there are different constitutive functions to describe water retention curve and hydraulic conductivity relation, for example, van Genuchten [12], Campbell [21], and Haverkamp *et al.,* [22]. The experimental dataset described by a constitutive function used in Richards' equation will be affecting the water infiltration profile.

The different water infiltration profiles resulted by different constitutive functions is a well-known problem, but there is still no published record on the solution. This problem is common when one is applying published coefficients of a particular water retention curve equation, for example Haverkamp [22], to conduct simulation study, and then, attempt to change between constitutive function equations, for example, from Haverkamp *et al.,* [22] to van Genuchten [20] equation. One would have recreated the dataset using Haverkamp equation coefficients first and then, redo the curve fitting on the dataset using Genuchten equation, hence, resulting different water infiltration profiles. Therefore, the first objective of the study is to provide a solution for the transition between constitutive equations and be able to produce similar water infiltration profile.

Richards' equation as described by Hopmans [17], it is a mass conservation equation and include the expression of Darcy law. An in-depth description on Richards' equation is also given by Germann [23] from the perspective of Navier-Stokes equation. However, take the case of water infiltration as an example, the influence of the coefficients ($D(\psi_m)$, K, $\theta_L - \psi_m$) as in Eq. (3) on water infiltration profile have not been discussed before. The second objective of this study is to reveal mechanics behind the Richards' equation as governed by the coefficients.

2. Methodology

Richards' equation was reported as not behaving as mass conserving equation when using Eq. (1) [24]. Eqs. (2) and (3) were free from the mass conservation problem. The water-based Richards' equation as in Eq. (3) was chosen and used in this study because the volumetric water content appeared in the independent and dependent variables which would ease the programming effort. The $D(\psi_m)$ is the soil water diffusivity (m$^2 \cdot$s^{-1}) and it is given by,

$$D(\psi_m) = K(\psi_m)\frac{\partial \psi_m}{\partial \theta_L} \tag{4}$$

where $\dfrac{\partial \psi_m}{\partial \theta_L}$ is the derivative of matric suction (ψ_m, -m) with respect to volumetric water content (θ_L, m$^3 \cdot$m^{-3}), and $K(\psi_m)$ is the hydraulic conductivity (m$\cdot$s^{-1}) of the soil.

The constitutive functions from Haverkamp *et al.,* [22] for the water retention curve ($\theta_L - \psi_m$) and hydraulic conductivity ($K(\psi_m)$) are given by,

$$\psi_m = -10^{-2}\exp\left[\frac{\alpha(\theta_s - \theta_r)}{\theta_L - \theta_r} - \alpha\right]^{1/\beta} \tag{5}$$

$$K = K_s\frac{A}{A + (-100\psi_m)^B} \tag{6}$$

Van Genuchten [20] water retention curve and hydraulic functions are given below,

$$\psi_m = -\frac{1}{\alpha_1}\left[\left(\frac{\theta_s - \theta_r}{\theta_L - \theta_r}\right)^{1/m} - 1\right]^{1/n} \tag{7}$$

$$K = K_s\left(\frac{\theta_L - \theta_r}{\theta_s - \theta_r}\right)^L\left\{1 - \left[1 - \left(\frac{\theta_L - \theta_r}{\theta_s - \theta_r}\right)^{1/m}\right]^m\right\}^2 \tag{8}$$

where α, α_1, A, B, β, L and n are fitting parameters, θ_s is saturated water content (m$^3 \cdot$m^{-3}), θ_r is residual water content (m$^3 \cdot$m^{-3}), K_s is saturated hydraulic conductivity (m$\cdot$s^{-1}), and m is also fitting parameter where $m = 1 - 1/n$.

To implement Richards' equation, Eq. (3) was discretized into algebraic equations using cell-centered finite-difference method, as follows,

$$\frac{\theta_{L(k)}{}^{n+1} - \theta_{L(k)}{}^{n}}{\Delta t} = \frac{K_{k+1/2}\left(\partial \psi_m / \partial \theta_L\right)_{k+1/2}}{\Delta z_k \left(0.5\Delta z_{k+1} + 0.5\Delta z_k\right)}\left(\theta_{L(k+1)}{}^{n+1} - \theta_{L(k)}{}^{n+1}\right)$$

$$- \frac{K_{k-1/2}\left(\partial \psi_m / \partial \theta_L\right)_{k-1/2}}{\Delta z_k \left(0.5\Delta z_k + 0.5\Delta z_{k-1}\right)}\left(\theta_{L(k)}{}^{n+1} - \theta_{L(k-1)}{}^{n+1}\right) \tag{9}$$

$$- \frac{K_{k+1/2}\vec{k} - K_{k-1/2}\vec{k}}{\Delta z_k}$$

where k indicates a center cell in z-direction of a cartesian coordinate system, $+1/2$ refers to the interface of cell located between cell k and $k+1$, $-1/2$ refers to cell interface between cell $k-1$ and k, n indicates the current time, $n+1$ indicates new time, and Δ refers to the difference of a parameter. Eq. (9) is oriented positive downward, so does Eq. (3). Complete description of the numerical scheme was explained in Goh and Noborio [25]. The equations were coded in Fortran.

Water infiltration phenomenon was used as a case study. The soil type used was Yolo light clay. A schematic diagram as in Figure 1(a) provides a simplistic view of the model environment.
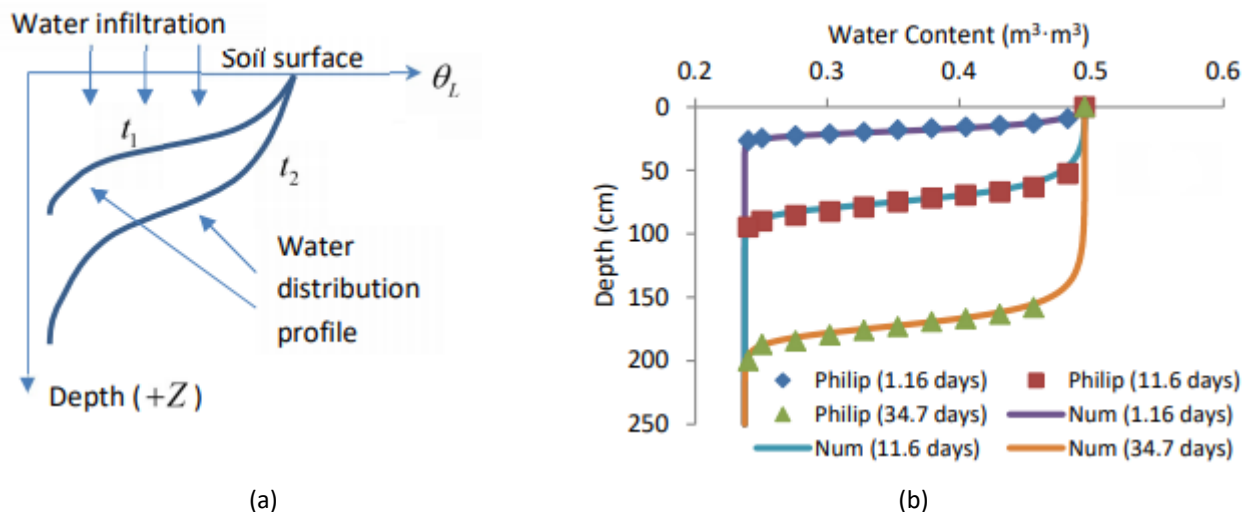

Fig. 1. Water distribution in the soil by: (a) a schematic diagram for soil water infiltration profile, and (b) water infiltration profile in Yolo light clay after 1.16, 11.6 and 34.7 days. Philip and Num referred to Philip's semianalytical solution and the Fortran coded numerical simulation, respectively

3. Results
3.1 Validation of Simulation Results with Philip's Semianalytical Solution

The Yolo light clay soil was initially wet with homogeneous soil moisture content of 0.2376 $m^3 \cdot m^{-3}$. The clay properties are shown in Table 1. The water infiltration begins with the upper soil body saturated with 0.495 $m^3 \cdot m^{-3}$ soil moisture content, and the lower soil body was assumed permeable to water, that is to say, water is free to move in and out. The total soil depth was taken as 2.5 m. The soil depth was further divided into 100 cells, which each cell represents 2.5 cm. Each cell was governed by Eq. (9) and the cells were simulated simultaneously in Fortran.

To make sure Richards' equation was programmed correctly, the simulation has to be validated either by experimentation or analytical solution dataset. In this case study, we used Philip's semianalytical solution [26-28] to validate the Fortran code. The results are shown in Figure 1(b). The

R-squared showed a good match between Fortran simulation and the analytical solution of water infiltration profiles at 1.16, 11.6 and 34.7 days that were 0.9992, 0.9988 and 0.9994, respectively.

Table 1
The soil type is Yolo light clay. The parameters used were retrieved from Haverkamp *et al.,* [22]

Parameters						
α	θ_r (m^3·m^{-3})	θ_s (m^3·m^{-3})	β	A	B	K_s (m·s^{-1})
739	0.124	0.495	4	124.6	1.77	1.23×10^{-7}

Note: α, A, B and β are fitting parameters, θ_s is saturated water content, θ_r is residual water content, and K_s is saturated hydraulic conductivity.

3.2 The problem of Transition Between Constitutive Equations

Water retention curve is also known as water characteristic curve. There are many water characteristic curve equation [29], and new equation is continuously being proposed [30]. Hence, it would be impossible to address all the transition issues between all the equations. However, a simple case study would be discussed here as an example to demonstrate the technique used, which can be referred to as a transition method in solving similar problem.

Changing the characteristic curve equation, for example from Haverkamp to Genuchten equations, with the same dataset could result in different water infiltration profile, as showed in Figure 2(a). As shown in the Figure 2(a), at 34.7 days, van Genuchten equation simulation had under predicted the water infiltration profile. In achieving objective one that is to have an equal water infiltration profiles that transition from Haverkamp to van Genuchten equations, Eqs. (7) and (8) were modified to,

$$\psi_m = -\frac{1}{\alpha_1}\left[\left(\frac{\theta_s - \theta_r}{\theta_L - \theta_r}\right)^{1/m_1} - 1\right]^{1/n_1} \tag{10}$$

$$K = K_s\left(\frac{\theta_L - \theta_r}{\theta_s - \theta_r}\right)^L \left\{1 - \left[1 - \left(\frac{\theta_L - \theta_r}{\theta_s - \theta_r}\right)^{1/n_2}\right]^{m_2}\right\}^2 \tag{11}$$

where m_1, n_1, m_2, and n_2 are fitting parameters. The curve-fitting parameters were increased from 3 to 6. The intention of the modification was to increase the ability of the Eqs. (10) and (11) to have a better curve-fitting ability than Eqs. (7) and (8). Figure 2(b) has proven the effectiveness of this method which was able to recreate the center of the water infiltration front, possible beyond 34.7 days.

Therefore, the best practice is to make the equation parameters unbound between the characteristic curve equation (Eq. (10)) and the hydraulic conductivity equation (Eq. (11)). This practice can be applied to other studies without much difficulty, however, most of the software, for example HYDRUS, does not allow such flexibility in the equation setting.
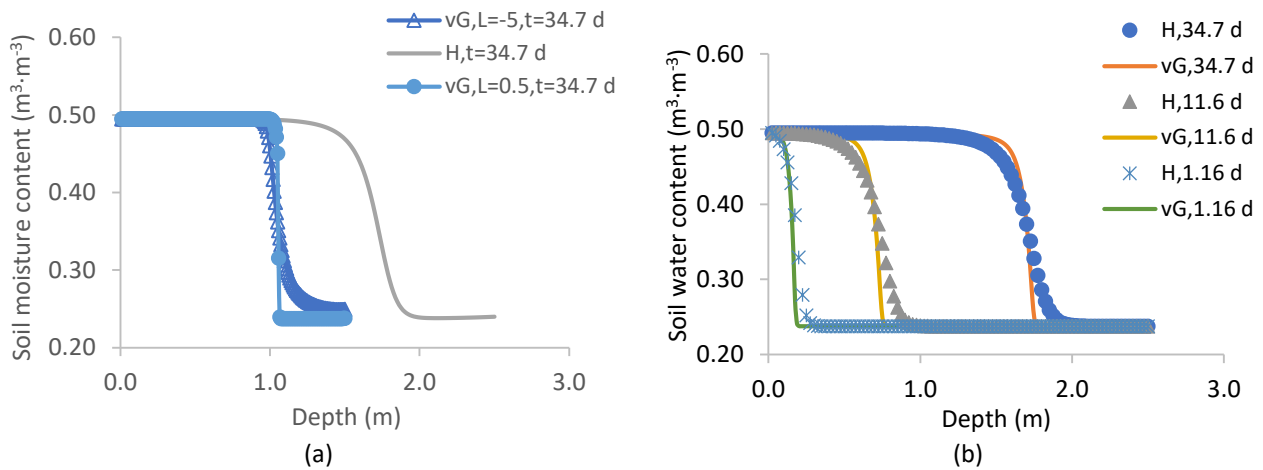
Fig. 2. Water infiltration profile for Genuchten and Haverkamp equations simulated at: (a) 34.7 days with m, n, α_1 curve-fitted, while L fixed at 0.5 and -5; and (b) 1.16, 11.6, 34.7 days with L, m_1, n_1, m_2, n_2, α_1 curve-fitted. Note: vG and H corresponding to Genuchten and Haverkamp

3.3 The mechanics of Richards' Equation

Figure 2(b) showed that Richards' equation using improved Genuchten Eqs. (10) and (11) has significantly improved the simulation to match with those of the Haverkamp. The water infiltration front appeared sharply from 34.7 to 1.16 days of simulation. Although the improved Ganuchten is able to reproduce the waterfront center, Genuchten equation reproduces sharper waterfront than that of Haverkamp. The Genuchten waterfront appeared extrude in the high-water content region and intrude at the low water content region. This can be explained by the relation between hydraulic conductivity (K) and soil moisture content, as in Figure 3(a), where the Genuchten curve is extruded in high water content region and intrude at the other region. Figure 3(b) shows the rarely reported relation between $\dfrac{\partial \psi_m}{\partial \theta_L}$ and the soil moisture content. The graph shows a concave curve with a minimum was reached at 0.4586 m³·m⁻³, which indicates a minimum gradient. Known that $D(\theta_L)$ in Eq. (4) consist of $K(\theta_L)$ and $\dfrac{\partial \psi_m}{\partial \theta_L}$, the former variable is always high at high soil moisture content as in Figure 3(a), whereas the latter variable would have to depend on the region soil wetness. This means that $\left(\partial \psi_m / \partial \theta_L\right)_{k-1/2} - \left(\partial \psi_m / \partial \theta_L\right)_{k+1/2}$ in space may not be always positive in value because it depends on the level of soil wetness. Whereas the $D(\theta_L)_{k-1/2} - D(\theta_L)_{k+1/2}$ is always positive for water infiltration front as long as the water is continuously infiltrating into the soil, which explain the increasing soil water content with time in the soil depth.
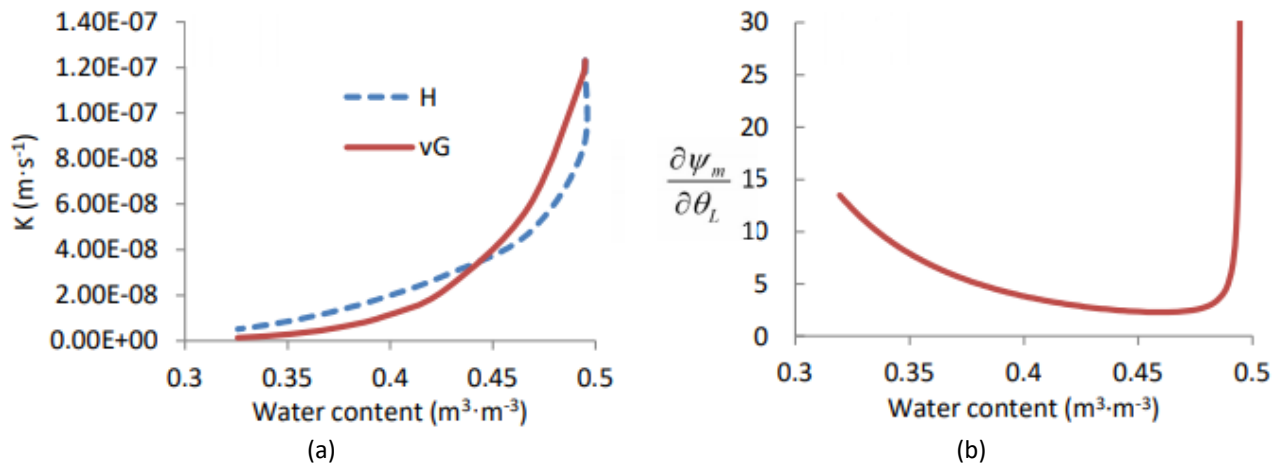
Fig. 3. (a) Hydraulic conductivity varies with soil moisture content, and (b) the derivative of soil matric suction with respect to moisture content varies with soil moisture content

4. Conclusions

Richards' equation is important to govern water flow in the unsaturated soil, which acts as an interface between the atmosphere and the groundwater. Richards' equation which is frequently used in conjunction with other constitutive equations (water characteristic curve and hydraulic functions) does not allow easy transition between equations, for example from Haverkamp to Genuchten. In the first objective, we have demonstrated that an easy modification of the existing Genuchten variables was able to reproduce water infiltration front. In the second objective, we have discussed the importance of understanding the variable in the Richards' equation in order to be able to understand the mechanics of water transport and also be able to describe the observed sharp water infiltration front produced from using Genuchten equation. The hidden property of the derivative of soil matric suction with respect to the soil moisture content is often overlooked has influenced on the influx of water into the soil.

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