Penerbit Akademia Baru

Journal of Advanced Research Design

Journal homepage:
https://akademiabaru.com/submit/index.php/ard
ISSN: 2289-7984

Adaptive Hybrid Reduced Differential Transform Method in Solving Nonlinear Schrodinger Equations

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ARTICLE INFO

Article history:
Received 4 November 2024
Received in revised form 17 December 2024
Accepted 7 April 2025
Available online 30 April 2025

Keywords:
Adaptive multistep differential transform method; multistep reduced differential transform method; nonlinear Schrodinger equations; adaptive algorithm; Adomian polynomials

ABSTRACT

In this paper, piecewise-analytical and numerical solutions are obtained by a new adapted approach named the Adaptive Hybrid Reduced Differential Transform Method (AHRDTM). The fundamentals of the method are introduced, followed by its application to Nonlinear Schrodinger Equations (NLSEs). Analytical and numerical solutions are acquired using piecewise convergent series with computationally feasible components across a sequence of sub-intervals of varying length. This succeeds due to the adaptive algorithm introduced and the substitution of the non-linear term in the NLSEs with their corresponding Adomian polynomials. The accuracy of the AHRDTM is observed through numerical comparisons between the proposed method and the Modified Reduced Differential Transform Method (MRDTM) with their respective exact solutions. The absolute errors presented in the table reveal that the proposed approach has better accuracy in solving the considered equations. The results and pictorial illustrations have been provided to demonstrate the reliability of the method's accuracy in obtaining approximate solutions.

1. Introduction

The partial differential equation (PDE) is a crucial type of differential equation (DE) employed to elucidate and simulate scientific phenomena in various fields such as optics, acoustics, fluid mechanics, hydrodynamics and astronomy [1,2]. The significance of this concept and its practical applications extends to both pure and modern mathematical research areas [3-5]. Given the intricate nature of problem-solving for PDEs, numerous methods have been devised to solve such challenges [6-8]. Analytical techniques like the Optimal Homotopy Asymptotic Method (OHAM) [9], φ^6-Model Expansion Method [10] and Modified Extended Direct Algebraic Method (MEDAM) [11] are employed to tackle these complex models due to their inherent structural intricacies. Some effective approaches in PDE solutions include the Quadratic Jacobi's Elliptic Function Expansion Method [12],

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https://doi.org/10.37934/ard.129.1.3345

Improved Shooting Method [13], Exponential Time Differencing Runge–Kutta Scheme [14] and Painlevé Analysis [15].

Several hyperbolic wave-type equations, including the Korteweg-De Vries Equations (KdVEs), Klein-Gordon Equations (KGEs), Schrodinger Equations (SEs), Harry Dym Equations and Burgers Equations have been utilized as mathematical models for various types of waves. This paper focuses on the Nonlinear SEs (NLSEs), a well-known equation with applications in hydrodynamics, plasma waves, nonlinear optical waves, oceanography, deep surface water waves, ocean roughness, biology, quantum mechanics and light emission in cables of fibre optics [16]. The NLSEs have been studied both numerically and analytically using diverse methods, including the Darboux–Bäcklund Transformation [17], the Cubic Exponential B-Spline Collocation Method (CuEBCM) [18], Modified F-Expansion Method [19] and various other methods [20-22]. Approximate analytical solutions, also known as semi-analytical methods, are valuable in solving NLSEs. One such method is the Q-Homotopy Analysis Transform Method (q-HATM), which has been applied to obtain analytical and approximate solutions of NLSE with higher dimension by Akinyemi *et al.,* [23]. Alshammari *et al.,* [24] utilized the Adomian Decomposition Transform Method (ADTM) and the Variational Iteration Transform Method (VITM) to solve fractional system Jaulent–Miodek equations connected with energy dependent Schrodinger potential. Another significant analytic method employed in this paper is the Differential Transform Method (DTM). In DTM, the provided differential equation, along with the initial conditions, is transformed into a recursive equation, ultimately yielding the solution as a Taylor series. DTM is the original method used in this study.

Scientists have brought attention to the DTM and its advanced variants. In 2009, Keskin *et al.,* [25,26] introduced the Reduced DTM (RDTM) as a semi-analytical method to solve problems related to PDEs. This method gained significant traction among researchers for its effectiveness in solving a variety of problems [27-30]. However, challenges arose when dealing with complex PDEs.

Recognizing the complexities encountered in solving fractional Korteweg-De Vries Equations (KdVEs), Saha Ray [31] proposed a modification to the fractional RDTM. This modification involves replacing the nonlinear term with Adomian polynomials, simplifying the derivation of solutions and reducing the number of calculated terms. Consequently, this adaptation of the RDTM is known as the Modified RDTM (MRDTM) and proves to be a valuable approach for handling problems with highly nonlinear terms.

The Multistep DTM (MsDTM) represents another advancement in DTM. Initially presented in 2010 by Odibat *et al.,* [32] this semi-analytic method has been applied to various systems. MsDTM generates a solution with a rapidly converging convergent series, particularly over a sequence of equal-length subintervals, thereby enhancing the overall convergence of the series solution. Building upon this, researchers such as Al-Smadi *et al.* [33] and Momani *et al.,* [34] applied the multistep scheme to the RDTM, resulting in the Multistep RDTM (MsRDTM) for solving fractional PDEs [35].

In 2018 and 2019, Che Hussin *et al.,* [36,37] introduced and implemented the Multistep Modified RDTM (MMRDTM) by combining the concepts of MsRDTM and MRDTM. This approach has proven effective in solving NLSEs and fractional NLSEs (FNLSEs). Additionally, Sabdin *et al.,* [38,39] have applied the MMRDTM in solving nonlinear telegraph equations (NLTEs) and time fractional NLTEs (TFNLTEs) with source terms.

However, a notable limitation of employing a multistep scheme is its reduced efficiency when dealing with large intervals. Therefore, an adaptive approach becomes crucial to address such challenges, allowing for the incorporation of variable-length step sizes. This adaptability is vital for ensuring convergence over a significant time frame across a sequence of subintervals with varying lengths. Recognizing this need, two distinct adaptive algorithms with different approaches were developed by Gökdoğan *et al.,* [40] and El-Zahar [41].

Building upon the mentioned references, both adaptive schemes applied to the DTM have been termed the Adaptive MsDTM (AMsDTM). These adaptive approaches have been implemented to obtain approximate analytical solutions for nonlinear problems [42].

The objective of this paper is to introduce a novel approach named the Adaptive Hybrid Reduced Differential Transform Method (AHRDTM) for solving the NLSE. The term "Adaptive Hybrid" signifies the integration of the adaptive multistep approach from AMsDTM by El-Zahar [41] and the modification of RDTM (MRDTM) through the use of appropriate Adomian polynomials. The adaptive algorithm is employed to obtain solutions with variable-length subintervals. The purpose is to observe the accuracy of the proposed method in solving considered NLSEs. The accuracy is observed by calculating the absolute error of AHRDTM and MRDTM opposing their respected exact solutions. The outcomes show that AHRDTM has greater accuracy than the MRDTM. The results indicate the reliability of the new approach in solving the considered problems.

The remaining sections of this work are organized as follows: Section 2 discusses definitions, the adaptive algorithm and solution formulations. Section 3 illustrates the application of AHRDTM in various NLSEs, presenting solutions through tables and graphical illustrations. Finally, Section 4 provides concluding observations.

2. Development of Adaptive Hybrid Reduced Differential Transform Method

This innovative method is a synthesis of techniques from the Reduced Differential Transform Method (RDTM), Modified Reduced Differential Transform Method (MRDTM), Multistep Reduced Differential Transform Method (MsRDTM) and Adaptive Multistep Differential Transform Method (AMsDTM). The specific steps or limitations associated with each of these methods are outlined below.

2.1 Reduced Differential Transform Method

A two-variable function $u(x,t)$ is under consideration and it can be expressed as the product of two single-variable functions: $u(x,t) = f(x)g(t)$. Leveraging the foundational properties of the one-dimensional differential transform, the function $u(x,t)$ can be represented as the Eq. (1) below,

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{1}$$

In this context, the t-dimensional spectrum function of $u(x,t)$ is represented as $U_k(x)$. The essential definitions of the RDTM are outlined as follows [25]:

i. Definition 1: If the domain of interest's function $u(x,t)$ is analytical and continuously differentiable with regard to time t and space x as in Eq. (2),

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}, \tag{2}$$

Where, the transformed function is the t-dimension spectrum function $U_k(x)$. In this paper, the primary function is denoted by the small letter $u(x,t)$, while the altered function is symbolized by the capital letter $U_k(x)$.

ii. Definition 2: Given the following for the differential inverse transform of $U_k(x)$ as in Eq. (3),

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{3}$$

Then, by combining Eqs. (2) and (3), we obtain Eq. (4),

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} t^k. \tag{4}$$

Building upon the definitions provided earlier, the concept of RDTM is derived from the expanded power series. Consider the following operator-form nonlinear PDE to explain the fundamental RDTM concepts as in Eq. (5):

$$\mathcal{L}u(x,t) + \mathcal{R}u(x,t) + \mathcal{N}u(x,t) = g(x,t), \tag{5}$$

with initial condition of Eq. (6),

$$u(x,0) = f(x), \tag{6}$$

Where, $\mathcal{L} = \frac{\partial}{\partial t}$, $\mathcal{R}$ is a partial derivatives linear operator, $\mathcal{N}u(x,t)$ is a nonlinear operator and $g(x,t)$ is an inhomogeneous term. Based on AHRDTM, the iteration formula shown below may be formed as in Eq. (7),

$$(k+1)U_{k+1}(x) = \mathcal{G}_k(x) - \mathcal{R}U_k(x) - \mathcal{N}U_k(x), \tag{7}$$

Where, $U_k(x), \mathcal{R}U_k(x), \mathcal{N}U_k(x)$ *and* $\mathcal{G}_k(x)$ are the transformations of the respected functions $\mathcal{L}u(x,t), \mathcal{R}u(x,t), \mathcal{N}u(x,t),$ *and* $g(x,t)$. We write the transformed initial condition Eq. (6)

$$U_0(x) = f(x). \tag{8}$$

2.2 Modified Reduced Differential Transform Method

Ray [31] denoted the nonlinear term as follows in Eq. (9),

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n\big(U_0(x), U_1(x), \dots, U_n(x)\big). \tag{9}$$

The method proposed for computing the Adomian polynomials as shown in Eq. (10), and Eq. (11),

$$A_0 = N\big(U_0(x)\big), \tag{10}$$

$$A_n(U_0(x), U_1(x), \dots, U_n(x)) = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}[N(\textstyle\sum_{k=0}^{n} \lambda^k U_k(x))]\right)_{\lambda=0}, n \geq 1, \tag{11}$$

such that $\mathcal{N}U_k(x)$ is the term of nonlinearity. When the nonlinear term is replaced by its Adomian polynomial, the solution yields Eq. (12),

$$(k+1)U_{k+1}(x) = \mathcal{G}_k(x) - \mathcal{R}U_k(x) - A_k. \tag{12}$$

It's important to note that this approach doesn't necessitate time-consuming computations involving high derivatives. Iterative calculation can be used to obtain $U_k(x)$ values by combining Eq. (8) into Eq. (12). Furthermore, the set of inverse transformation values, $\{U_k(x)\}_{k=0}^{n}$ yields the approximate solution as follows in Eq. (13),

$$u_n(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \,, t \in [0,T]. \tag{13}$$

2.3 Multistep Reduced Differential Transform Method

The MsRDTM identifies the RDTM series by partitioning the interval $[0,T]$ into R subintervals of equal length. However, dealing with both linear and nonlinear differential equations often requires choosing a small-time step size. This, in turn, leads to obtaining RDTM solutions across a greater number of subintervals. Therefore, selecting a smaller time step size and increasing the number of subintervals result in longer computation times. To tackle this challenge, a novel approach is needed and an adaptive methodology is suggested as a solution.

2.4 Adaptive Multistep Differential Transform Method

The following adaptive time step-size control algorithm is taken from the algorithm introduced in AMsDTM. The algorithm is then applied according to the adaptive scheme by El-Zahar *et al.,* [41]:

i. One gives the admissible local error $\delta > 0$ and chooses the order N of the multistep scheme.
ii. From calculations, the values $|U_{k,r}(N)|$, $k = 1,2,\ldots,n$, are known for every solution component k.
iii. At the grid point t_r, we calculate the value $E_N = max(|U_{k,r}(N)|)$, $k = 1,2,\ldots,n$.
iv. We select the step-size h_r for which $h_r = \tau \left(\frac{\delta}{E_N}\right)^{\frac{1}{N}} \leq h_{max}$ and $t_{r+1} = t_r + h_r$, where τ is a safety factor and h_{max} is the maximum allowed step-size.

Firstly, MRDTM is applied to the initial value problem of interval $[0,t_1]$. Then, by using the initial conditions as in Eq. (14),

$$u(x,0) = f_0(x). \tag{14}$$

We obtain the approximate result in Eq. (15),

$$u_1(x,t) = \sum_{k=0}^{\mathrm{k}} U_{k,1}(x)t^k \,, t \in [0,t_1]. \tag{15}$$

The adaptive step-size control algorithm is then applied to determine t_1 of $[0,t_1]$. Then, for each subinterval $[t_{r-1},t_r]$, the initial condition in Eq. (16),

$$u_r(x,t_{r-1}) = u_{r-1}(x,t_{r-1}), \tag{16}$$

is used for $r \geq 2$ and the implementation of AHRDTM to the initial value problem on $[t_{r-1},t_r]$. For $r = 1,2,\ldots,R$, the algorithm is then applied repeatedly for each subinterval $[t_{r-1},t_r]$ of R variable-length subintervals to determine each t_r. Thus, the interval $[0,T]$ is a combination of variable-length subintervals, $[t_{r-1},t_r]$ for $t \in [0,T]$.

The process is iteratively repeated to construct an approximate solutions sequence $u_r(x,t)$ for $r = 1,2,\dots,R$, as follows in Eq. (17),

$$u_r(x,t) = \sum_{k=0}^{K} U_{k,r}(x)(t - t_{r-1})^k\,, t \in [t_{r-1}, t_r]. \tag{17}$$

Finally, the AHRDTM proposes the following solutions in Eq. (18),

$$u(x,t) = \begin{cases} u_1(x,t),\ for\ t \in [0, t_1] \\ u_2(x,t),\ for\ t \in, [t_1, t_2] \\ \vdots \\ u_R(x,t),\ for\ t \in, [t_{R-1}, t_R] \end{cases}. \tag{18}$$

It is crucial to note that when the step size $s = T$, the MRDTM is derived from AHRDTM.

3. Results

Three numerical examples have been examined to demonstrate the reliability of the AHRDTM and its efficacy in solving the NLSE:

i. Example 1: Cubic NLSE was taken consideration [43]:

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{19}$$

is considered with initial condition:

$$u(x,0) = e^{ix}. \tag{20}$$

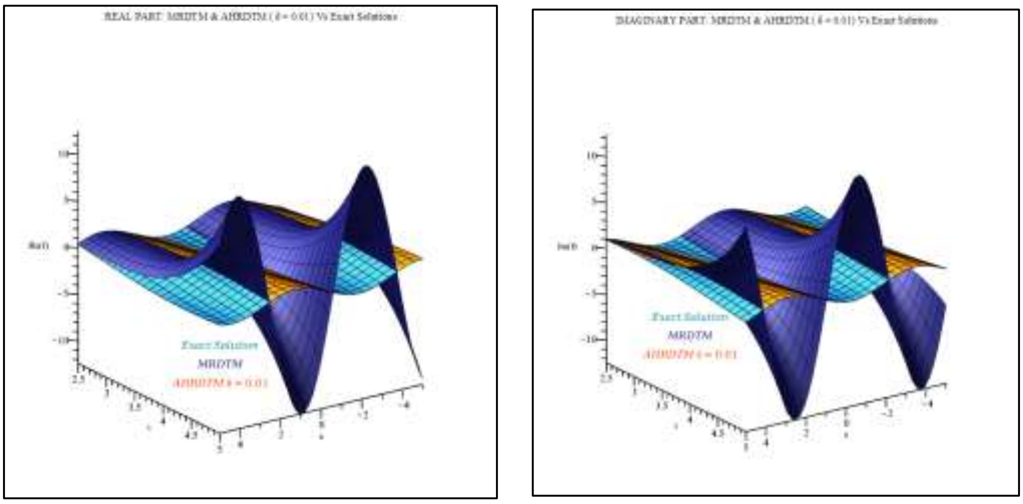
$e^{i(x+t)}$ is the exact solution of this equation. By applying the AHRDTM to Eq. (19) and using fundamental properties of AHRDTM, we have:

$$U_{k+1,r}(x) = \left(\frac{I}{k+1}\right)\left(\frac{\partial^2}{\partial x^2} U_{k,r}(x) + 2A_{k,r}\right). \tag{21}$$

We write the transformed initial condition Eq. (20) as:

$$U_0(x) = e^{ix}. \tag{22}$$

The adaptive algorithm is then implemented to obtain an approximate solution. Graphical comparisons of the approximate solution using MRDTM, AHRDTM at $N = 6$, with respect to the admissible local error, $\delta = 0.01$ and the exact solution for $t \in [2.5, 5]$ and $x \in [-5, 5]$, encompassing the real and imaginary parts, are presented in Figure 1(a) and Figure 1(b) respectively. The comparisons in Figure 1(a) and Figure 1(b) distinctly reveal that the graphs of AHRDTM with $\delta = 0.01$ closely resemble the shape and size of the exact solutions in contrast to the graph of MRDTM. Consequently, AHRDTM demonstrates superior accuracy in approximating solutions. The solutions obtained by AHRDTM for this type of NLSE are thus shown to be nearly accurate in comparison to the exact solutions.



(a) (b)

Fig. 1. The graphs shown in (a) and (b) are the exact solutions, MRDTM and AHRDTM with $\delta = 0.01$ involving the real and imaginary part respectively

Table 1 provides the performance error analysis. According to the table, numerical results for absolute error (AE) and error norms L_2 and L_∞ from AHRDTM with $\delta = 1.0 \times 10^{-10}$ are smaller, indicating greater accuracy than MRDTM. AHRDTM yields superior results in solving the NLSE compared to MRDTM.

Table 1
Error analysis of semi-analytic solution for MRDTM and AHRDTM

T	Exact Solutions	AE MRDTM	AE AHRDTM
0	0.5403023059 + 0.8414709848I	0	0
0.1	0.4535961214 + 0.8912073601I	$2.996664813 \times 10^{-10}$	$2.996664813 \times 10^{-10}$
0.2	0.3623577545 + 0.9320390860I	$2.580697580 \times 10^{-9}$	$2.580697580 \times 10^{-9}$
0.3	0.2674988286 + 0.9635581854I	$4.330415684 \times 10^{-8}$	$4.270316148 \times 10^{-8}$
0.4	0.1699671429 + 0.9854497300I	$3.246369357 \times 10^{-7}$	$4.715347283 \times 10^{-8}$
0.5	0.07073720167 + 0.9974949866I	$1.547822690 \times 10^{-6}$	$5.406933419 \times 10^{-8}$
0.6	-0.02919952230 + 0.9995736030I	$5.542108513 \times 10^{-6}$	$9.859808365 \times 10^{-8}$
0.7	-0.1288444943 + 0.9916648105I	$1.629145204 \times 10^{-5}$	$1.073450511 \times 10^{-7}$
0.8	-0.2272020947 + 0.9738476309I	$4.144872897 \times 10^{-5}$	$1.186779255 \times 10^{-7}$
0.9	-0.3232895669 + 0.9463000877I	$9.443486255 \times 10^{-5}$	$1.674857009 \times 10^{-7}$
1.0	-0.4161468365 + 0.9092974268I	$1.972130480 \times 10^{-4}$	$1.808347588 \times 10^{-7}$
	L_2	$2.022634535 \times 10^{-4}$	$3.210198349 \times 10^{-7}$
	L_∞	$1.972130480 \times 10^{-4}$	$1.808347588 \times 10^{-7}$

ii. <u>Example 2:</u> NLSE with zero trapping potential of the form [43]:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u = 0, \tag{23}$$

was considered with initial condition:

$$u(x,0) = e^{ix}. \tag{24}$$

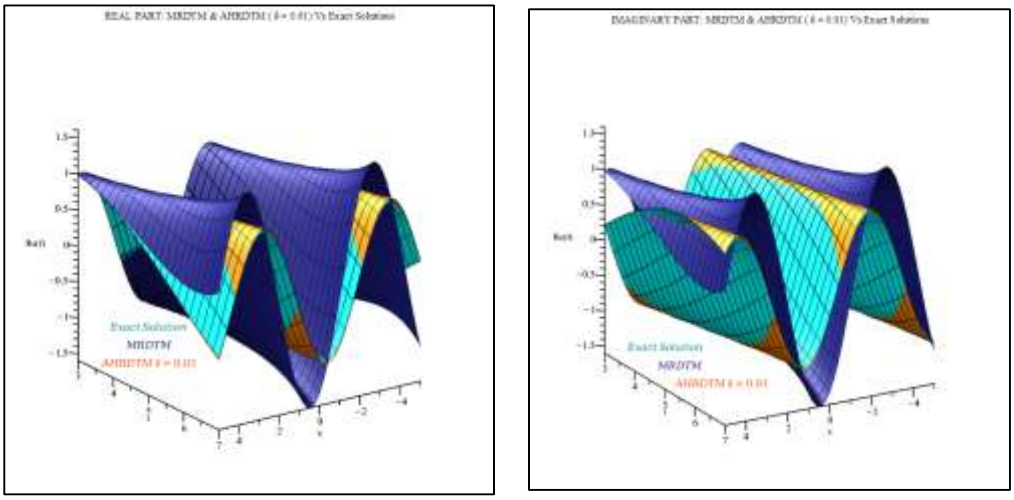
$e^{i(x+\frac{t}{2})}$ is this equation's exact solution. By applying the AHRDTM to Eq. (23) and using fundamental properties of AHRDTM, we have:

$$U_{k+1,r}(x) = \left(\frac{I}{k+1}\right)\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}U_{k,r}(x) + A_{k,r}\right). \tag{25}$$

We write the transformed initial condition Eq. (24) as:

$$U_0(x) = e^{ix}. \tag{26}$$

Next, the adaptive algorithm is applied to obtain an approximate solution for this example. Graphical comparisons of the approximate solution using MRDTM, AHRDTM at $N = 6$, with respect to the specified tolerance, $\delta = 0.01$ and the exact solution for $t \in [3, 7]$ and $x \in [-5, 5]$, encompassing the real and imaginary parts, are depicted in Figure 2(a) and Figure 2(b) respectively. The comparisons in Figure 2(a) and Figure 2(b) show that the graphs of AHRDTM with $\delta = 0.01$ closely resemble the shape and size of the exact solutions in contrast to the graph of MRDTM. Consequently, the solutions obtained by AHRDTM for this type of NLSE are demonstrated to be approximately accurate compared to the exact solutions.



(a) (b)

Fig. 2. The graphs shown in (a) and (b) are the exact solutions, MRDTM, AHRDTM with $\delta = 0.01$, which involve the real and imaginary part respectively

Table 2 presents the performance error analysis, including absolute error (AE) and error norms L_2 and L_∞. The numerical results from AHRDTM with $\delta = 1.0 \times 10^{-10}$ are notably more accurate than those from MRDTM, affirming that AHRDTM provides better results in solving the NLSE compared to MRDTM.

iii. <u>Example 3:</u> NLSE with trapping potential of the form [43]:

$$iu_t + \frac{1}{2}u_{xx} - ucos^2(x) - |u|^2 u = 0, \tag{27}$$

had been considered with initial condition:

$$u(x, 0) = \sin(x). \tag{28}$$

Table 2
Error analysis of semi-analytic solution for MRDTM and AHRDTM

T	Exact Solutions	AE MRDTM	AE AHRDTM
0	0.5403023059 + 0.8414709848I	0	0
0.1	0.4975710479 + 0.8674232256I	$3.551056181 \times 10^{-10}$	$3.551056181 \times 10^{-10}$
0.2	0.4535961214 + 0.8912073601I	$2.996664813 \times 10^{-10}$	$2.996664813 \times 10^{-10}$
0.3	0.4084874409 + 0.9127639403I	$2.441311123 \times 10^{-10}$	$2.441311123 \times 10^{-10}$
0.4	0.3623577545 + 0.9320390860I	$2.580697580 \times 10^{-9}$	$2.580697580 \times 10^{-9}$
0.5	0.3153223624 + 0.9489846194I	$1.204408569 \times 10^{-8}$	$1.204408569 \times 10^{-8}$
0.6	0.2674988286 + 0.9635581854I	$4.330415684 \times 10^{-8}$	$4.270316148 \times 10^{-8}$
0.7	0.2190066871 + 0.9757233578I	$1.276094040 \times 10^{-7}$	$4.499288833 \times 10^{-8}$
0.8	0.1699671429 + 0.9854497300I	$3.246369357 \times 10^{-7}$	$4.715347283 \times 10^{-8}$
0.9	0.1205027694 + 0.9927129910I	$7.404440897 \times 10^{-7}$	$4.938187927 \times 10^{-8}$
1.0	0.07073720167 + 0.9974949866I	$1.547822690 \times 10^{-6}$	$5.405972715 \times 10^{-8}$
	L_2	$1.751488817 \times 10^{-6}$	$1.076308023 \times 10^{-7}$
	L_∞	$1.547822690 \times 10^{-6}$	$5.405972715 \times 10^{-8}$

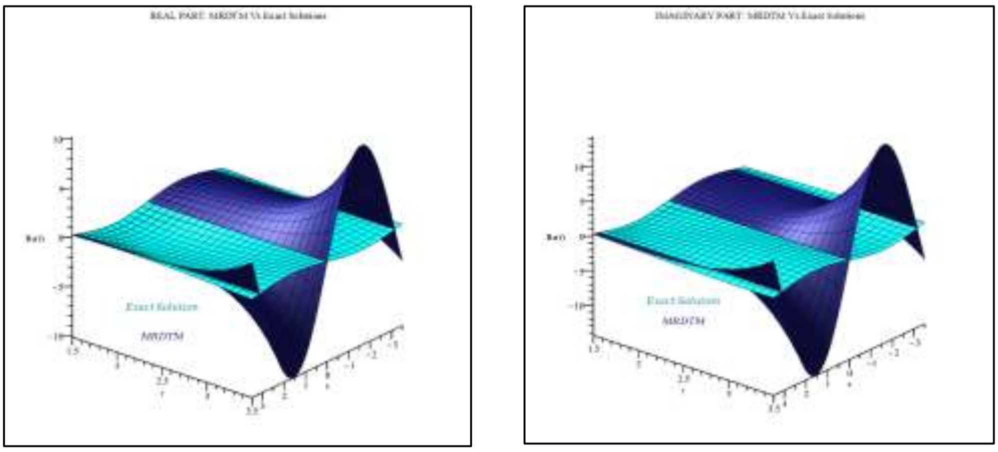
$\sin(x)e^{(-\frac{3i}{2}t)}$ is this equation's exact solution. By applying the AHRDTM to Eq. (27) and using fundamental properties of AHRDTM, we have:

$$U_{k+1,r}(x) = \left(\frac{I}{k+1}\right)\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}U_k(x) - U_k(x)cos^2(x) - A_{k,r}\right). \quad (29)$$

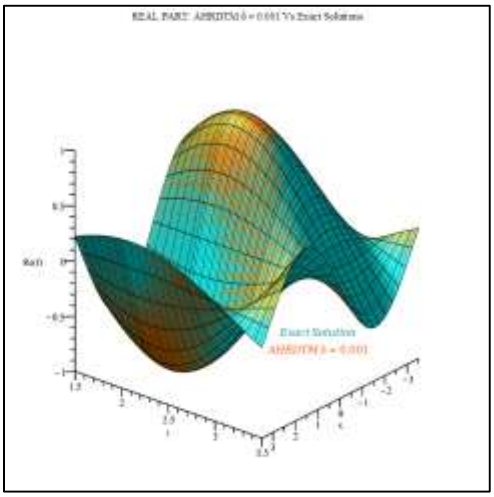
We write the transformed initial condition Eq. (28) as:

$$U_0(x) = \sin(x). \quad (30)$$

The adaptive algorithm is then applied to obtain an accurate approximate solution for this example. A graphical comparison of the approximate solutions using MRDTM, AHRDTM at $N = 6$, with respect to the specified tolerance, $\delta = 0.001$ and the exact solution for $t \in [1.5, 3.5]$ and $x \in [-3.5, 3.5]$, involving the real and imaginary parts, is depicted in Figure 3(a), Figure 3(b), Figure 3(c), Figure 3(d), Figure 3(e) and Figure 3(f). Figure 3(c) and Figure 3(d) illustrate that the graphs of AHRDTM with $\delta = 0.001$ closely and significantly resemble their exact solutions compared to the graph of MRDTM shown in Figure 3(a) and Figure 3(b). Consequently, AHRDTM demonstrates greater accuracy in approximating solutions. The solutions obtained by AHRDTM for this type of NLSE are shown to be significantly close to the exact solutions.



(a) (b)



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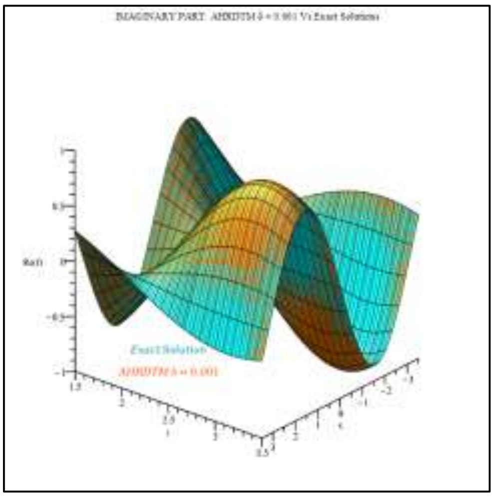

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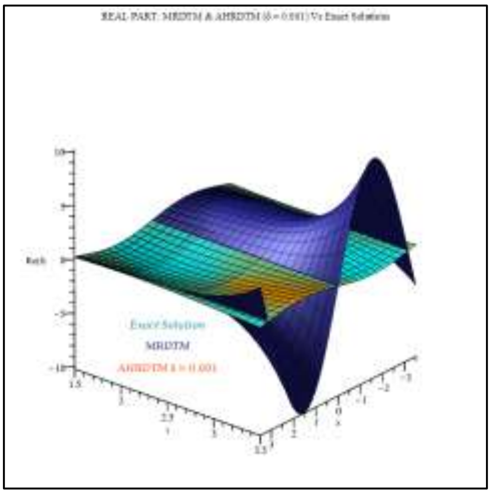
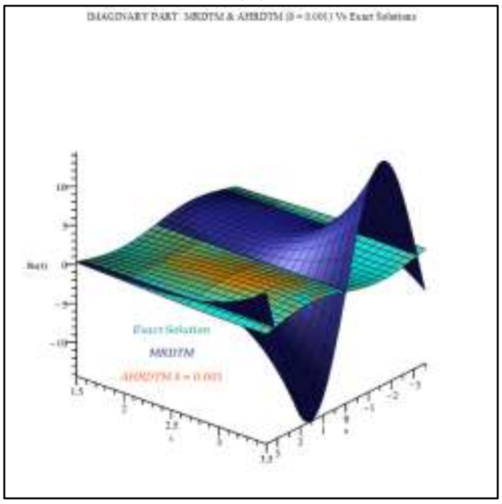


(e)



(f)

Fig. 3. The graphs shown in (a) and (b) are the exact solutions and MRDTM, (c) and (d) are the exact solutions and AHRDTM with $\delta = 0.001$, while (e) and (f) are the exact solutions, MRDTM and AHRDTM with $\delta = 0.001$, which involve the real and imaginary part, respectively

Table 3 the performance error analysis, where absolute error (AE) and error norms L_2 and L_∞ are utilized for error analysis. Numerical results from AHRDTM with $\delta = 1.0 \times 10^{-10}$ are noticeably more accurate than those from MRDTM. AHRDTM yields superior results in solving the NLSE compared to MRDTM.

Table 3
Error analysis of semi-analytic solution for MRDTM and AHRDTM

T	Exact Solutions	AE MRDTM	AE AHRDTM
0	0.8414709848	0	0
0.1	0.8320221727 - 0.1257478525I	$2.660964897 \times 10^{-10}$	$2.660964897 \times 10^{-10}$
0.2	0.8038879363 - 0.2486716794I	$3.644744902 \times 10^{-8}$	$3.644744902 \times 10^{-8}$
0.3	0.7577001100 - 0.3660108763I	$6.230741499 \times 10^{-7}$	$4.576620807 \times 10^{-8}$
0.4	0.6944959727 - 0.4751302582I	$4.663560160 \times 10^{-6}$	$7.734847669 \times 10^{-8}$
0.5	0.6156949531 - 0.5735792387I	$2.221039714 \times 10^{-5}$	$9.422267329 \times 10^{-8}$
0.6	0.5230667522 - 0.6591468660I	$7.946428659 \times 10^{-5}$	$1.221756010 \times 10^{-7}$
0.7	0.4186915997 - 0.7299114759I	$2.333622902 \times 10^{-4}$	$1.462022591 \times 10^{-7}$
0.8	0.3049135365 - 0.7842838476I	$5.930457850 \times 10^{-4}$	$1.711831216 \times 10^{-7}$
0.9	0.1842877727 - 0.8210428948I	$1.349438602 \times 10^{-3}$	$2.015161508 \times 10^{-7}$
1.0	0.05952330275 - 0.8393630887I	$2.814081292 \times 10^{-3}$	$2.244005936 \times 10^{-7}$
	L_2	$3.186381856 \times 10^{-3}$	$4.181551827 \times 10^{-7}$
	L_∞	$2.814081292 \times 10^{-3}$	$2.244005936 \times 10^{-7}$

4. Conclusions

This paper introduces a novel, efficient and accurate approach called the Adaptive Hybrid Reduced Differential Transform Method (AHRDTM), an adaptive approximation analytical method for handling NLSEs. The results illustrate the method's accuracy, effectiveness and reliability, as evidenced by the numerical outcomes and graphical representations. Consequently, AHRDTM emerges as a valuable mathematical tool for solving NLSEs, providing solutions with high accuracy and demonstrating a notable superiority over MRDTM in terms of accuracy. All calculations in this paper were conducted using Maple 2021.

Acknowledgement
This research is supported financially by the University of Malaysia Sabah through Research Grant UMSGreat GUG0576-1/2023, contributing to the successful advancement of the project.

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