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Theory-derived law of the wall for parallel flat-plates turbulent flow

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Abstract

It is well known that in a turbulent flow between two parallel flat plates, the horizontal mean velocity varies logarithmically with height (the so-called 'logarithmic-law-of-the-wall'). The law of the wall is a description of the mean velocity profile in wall bounded flows and has been regarded as one of the underpinning doctrine in the turbulence community for more than half a century. Much of our understanding in wall turbulence has been based from the continuum Navier-Stokes Equation (NSE). More recently, following studies of a modified Navier Stokes Equation, we applied a modified incompressible NSE to the flow of turbulent fluid between two parallel flat plates and solved it analytically [1]. We extended the analysis to the turbulent flow along a single wall and compared the results with the established controversial von Karman logarithmic law of the wall [2]. We found velocity profiles and velocity time evolution of a turbulent system, through simple numerical simulations, that cannot be reproduced from the classical NSE

Keywords: post-Navier Stokes equation; law of the wall; analytical solution; numerical simulations; flat-plate system

1. Introduction

For more than a hundred years, fluid turbulence has been has been one of the greatest mysteries of science. It has been an age-old topic of discussion among fluid dynamicists. Turbulence, as a state of fluid motion, has been understood before as something that is governed by dynamical laws such as the Navier-Stokes equation. Most fluid dynamicists agree that not only the problem of turbulence is still far from being solved, but also it is extremely difficult to agree on what is the problem to be solved [3].

The traditional method in standard hydrodynamics is to solve Navier- Stokes equation for stationary velocity profiles, which are parabolic for Poisseuille-Hagen flow in laminar regime and in good agreement with experiments. However, any agreement breaks down at the onset of turbulence as velocity profiles flatten and become non-stationary [4-5]. An explanation for the flattened velocity profiles was given by Prandtl and von Karman using the transverse component of the fluctuation of the velocity [5]. The faster molecules of the central region of the pipe show up in the boundary layer mix with slower molecules, and the velocity profile becomes roughly uniform except in the boundary layers.

For the description of turbulent flows, many renormalized perturbation theories all based on Reynolds equation were developed in the past [6]. The Reynolds equation, which aside from the mean velocity considers its arbitrary time fluctuations, like Navier-Stokes equation, ignores the structure of the molecules. Limitations of the Reynolds equation in the laminar-turbulent transition and its difficulty in explaining the origin of turbulence suggest study of the phenomenon directly from the Liouville equation.

It is known that difficulties arise when turbulence is studied from the standard Navier-Stokes equation alone. That was first observed by D. Ruelle [7]. One such recent effort modifies the Navier Stokes equation by a term resulting from a microscopic molecular consideration ignored by conventional hydrodynamics [1]. A quantum kick imparts a constant unit of momentum with certain probability per unit time to the molecules of the gas. This is a way of injecting energy into a system to see if turbulence results.

The law of the wall for the interior part of a wall-bounded turbulent shear flow is the cornerstone of fluid dynamics, and one of the very few pieces of turbulence theory whose outcome includes a simple analytic function for the mean velocity distribution, the logarithmic law. For wall-bounded flows, the so-called 'log law' is widely held to describe most turbulent wall-bounded flows, and lies at the heart of the most commonly used engineering computational models concerning turbulent flow in close proximity to surfaces. While there are several forms of the log law, the most common is the mean velocity profile normalized in variables given by $U^+ = \frac{1}{K}\ln y^+ + B_i$ where $U^+ = U / u_*$ and $y^+ = yu_* / v$. U is the mean velocity, y is the distance from the wall, v is kinematic viscosity and u_* is the friction velocity defined from the wall skin friction, τ_W, as $u_*^2 = \tau_W / \rho$ where ρ is the fluid density. The von Karman constant, K, and the additive constant B_i is widely thought to be universal constants. However, there seems to be little consensus on the values of these constants. For several decades, K was believed to be 0.41, but a few years ago Nagib estimated it to be 0.38 and K can be as high as 0.45 [9].

The history and theory supporting the universality of the log law for turbulent wall-bounded flows was examined by W. George. George articulated that the idea of a universal log law for wall-bounded flows is not supported either the theory or the data [9]. Several modifications of the log law of the wall for turbulent flow in smooth pipes are proposed already which are based from empirical data, one being a law consisting of three terms: a logarithmic term, a sine-square term and a cubic term [10]. Virtually most turbulence models are calibrated to reproduce the law of the wall in simple flows, such that when the law of the wall fails, current Reynolds-averaged turbulence models are not susceptible to failure [11].

Not long ago, attempts were made by several researchers such as Chorin and Barenblatt to describe the turbulent flow along a single plate by a power law [12]. They proposed that the velocity profile is not universal but a weakly varying power law with coefficients that vary with Reynolds Number, in general, of the form $U = Cy^\gamma$ where C and γ are Reynolds number dependent empirical constants, while von Karman log law is Reynolds number independent and is a good approximation for distances far enough from the wall only, the so called intermediate or overlap region. It fails in the regions near the wall, (viscous and buffer sublayers), and for very large distances from the wall. In the region near the wall, the viscous sublayer, the power law holds approximately for $\gamma = 1$, (linear dependency). Their analysis of the new experimental data adduces additional arguments against the von Karman-Prandtl universal logarithmic law and in favor of a specific power law. However, measurements done by Zanoun and Durst for high Reynolds numbers support the approximate validity of the log law in the intermediate or overlap region (inertial sublayer) [13].

This paper will present the equations derived from the modified Navier Stokes Equation which will be compared with known flow characteristics. We have applied flat plate boundary conditions to generate the respective velocity fields and corresponding velocity time evolution. The generated velocity profiles in the flat plate geometry will be explored and compared with experimental data. All of the flows we considered in these studies will be assumed to have undergone transition and is fully developed. The treatment of the equations will be paying attention entirely on smooth walls for which surface roughness have no quantifiable influence. The model reproduced the velocity profiles of turbulent flow between two flat plates. Extending the analysis to the flow along a single wall by letting the distance between two flat plates to go to infinity, we arrive at a result qualitatively similar to known results, nevertheless fundamentally different from the logarithmic law known to have discrepancies with empirical data by even 65% [2].

2. Application to a Flat-Plate System

We postulate a modified Navier-Stokes equation for the macroscopic velocity $\vec{U}$ of the form

$$\frac{\partial U_i}{\partial x} + U_j \partial_j U_i + \sigma U_i = -\frac{1}{\rho}\partial_j P_{ij} - \frac{\sigma \Pi}{m} \;\; ; \; i,j = 1,2,3 \tag{1}$$

where the last term of both sides of the equation are adopted from [1] to represent quantized kicks to the system, with σ as the probability per unit time that a particle of mass m is imparted a momentum kick Π. It can be interpreted as forcing by a paddle wheel which increases the velocity of a particle in a fluid. This is no longer a purely continuum model. We shall justify this ad-hoc approach from the apparent success of the model in explaining empirical data heretofore unexplained by continuum mechanics [4, 14]. All other terms follow conventional definitions. The justification of the strange terms appearing in the Equation 1 comes, besides its utility, as a result of inelastic interactions among the molecules of the fluid. Thus the contribution from the collision integral in Boltzmann transport equation used in their derivation does not reduce to zero unlike in the derivation of standard incompressible NS equation. From the hypothesis about the quantum nature of turbulence, dissipative effects are present from excitation to higher energy state of the particles due to inelastic collisions. It is well known that inelastic interactions among the molecules of the fluid result in deterministic chaos we associate with turbulence.

If one uses a decomposition of the pressure tensor P_{ij} into its diagonal and off-diagonal parts, $p\delta_{ij} + \sigma_{ij}, i, j = 1,2,3$, where p is the pressure and $\sigma_{ij} = \nu(\partial_i U_j - \partial_j U_i)$ is the component of shear stress tensor and applying incompressible fluid condition $\partial_j U_j = 0$, Equation 1 simplifies to

$$\frac{\partial U_i}{\partial t} + U_j \partial_j U_i + \sigma U_i = -\frac{\sigma \Pi}{m} - \frac{1}{\rho}(\partial_i p + \nu \partial_j{}^2 U_i) \tag{2}$$

where ν is the kinematic viscosity. For the flat plates configuration the mean velocity vector is $\bar{U} = (U(z,t), 0, 0)$ and we can further reduce it to Equation 3.

$$\frac{\rho}{\nu}\frac{\partial U}{\partial t} + kU - \frac{\partial^2 U}{\partial z^2} = g \tag{3}$$

The parameter $k = \dfrac{\rho\sigma}{\nu}$ is proportional to the probability of kicking particles to different momentum and $g = -\dfrac{1}{\nu}\partial_x p - \dfrac{k\Pi}{m}$ represents the constant effective pressure head per unit kinematic viscosity that drives the motion of the fluid. The solutions to the Equation 3 did not require numerical computations since it is easily solved analytically upon application of the necessary flow conditions. All plots were generated by substituting parameter values on the governing equation and adjusting the scaling of the plots to match with widely known flow profiles.

3. Results and discussion

3.1 New Law of the Wall

We find the stationary solution of Equation 3 using non-slip boundary conditions, $U(0,t)=U(L,t)=0$ and static initial condition, $U(z,0)=0$ and subject to flat plate boundary conditions, $U(0)=U(L)$, with plates at a distance L apart,

$$U(z)=\frac{g}{k}\left(1+\frac{\sinh\sqrt{k}(z-L)-\sinh\sqrt{k}z}{\sinh\sqrt{k}L}\right), k>0 \tag{4}$$

The results for the flat plate system are plotted in figure 1. Fixed parameters used for the numeric computations are: $g=100$ and $L=1$. For the numerical results for the two-plate system, it exhibits broad flattening of the velocity profiles, for large k values, for example with probability parameter $k=250$. It can be observed that for increasing k values, the velocity profile looks more like the predicted logarithmic curve from empirical studies. On the other hand, velocity profiles for lower values of the control parameter k are nearly parabolic or circular which are usually associated with laminar fluid flow. The flattening of the velocity profiles can be simply explained by increase in number of inelastic molecular interactions. With increased inelastic collisions, velocity profile exhibits flattening at the center which is characteristic of turbulence.

The velocity profile of the flow along a single flat plate, $z=0$, can be obtained simply by letting the distance, L, approach infinity. Then for the mean velocity of the flow we get

$$U_w(z)=\lim_{L\to\infty}\frac{g}{k}\left[1+\frac{\sinh\sqrt{k}(z-L)-\sinh\sqrt{k}z}{\sinh\sqrt{k}L}\right] \tag{5}$$

$$U_w(z)=\frac{g}{k}\left[1-\exp\left(-\sqrt{k}z\right)\right] \tag{6}$$

The result for the wall turbulence for a single wall flow is plotted in Figure 2, with the same parameters, except $L\to\infty$. For small distances z from the wall (boundary viscous sublayer), the exponential velocity profile reduces approximately to linear in good agreement with observations. The exponential description of the velocity profile deviates from the logarithmic law for small distances z, and is somehow consistent with an exponential profile derived using a field theoretical method in [8].

3.2 Velocity Time Evolution

A complete non-stationary solution of equation 3, the time development of the velocity profiles, can be found. If one writes a solution of equation 3 with $g=0$ as $u_G^{(1)}=Aexp\left(-\frac{vkt}{\rho}\right)+By(z)$ and another in the form $u_G^{(2)}=Cexp\left(-\frac{vkt}{\rho}\right)y(z)$ where A, B and C are constants to be determined. The solution of Equation 3 for $g\neq 0$, satisfying static and non-slip boundary conditions can be constructed as, $U(z,t)=\frac{g}{k}+u_G^{(1)}+u_G^{(2)}$. With

$y(z) = \left(\frac{\sinh\sqrt{k}z - \sinh\sqrt{k}(z-L)}{\sinh\sqrt{k}L} \right)$ and constants $A = B = -C = -\frac{g}{k}$ the time evolution of the profile is,

$$U(z,t) = \frac{g}{k}\left[1 - \exp\left(-\frac{vkt}{\rho}\right) - y(z)\right] + \frac{g}{k} y(z) \exp\left(-\frac{vkt}{\rho}\right) \tag{7}$$

The exponential term $\exp\left(-\frac{vkt}{\rho}\right)$ can be neglected as $t \to \infty$, for equation 7 to simplify into the equation 4 form. The same fixed parameters were used for the velocity profile progression plots in figure 3. It shows how the velocity profile approaches that for equation 4 for increasing t values.

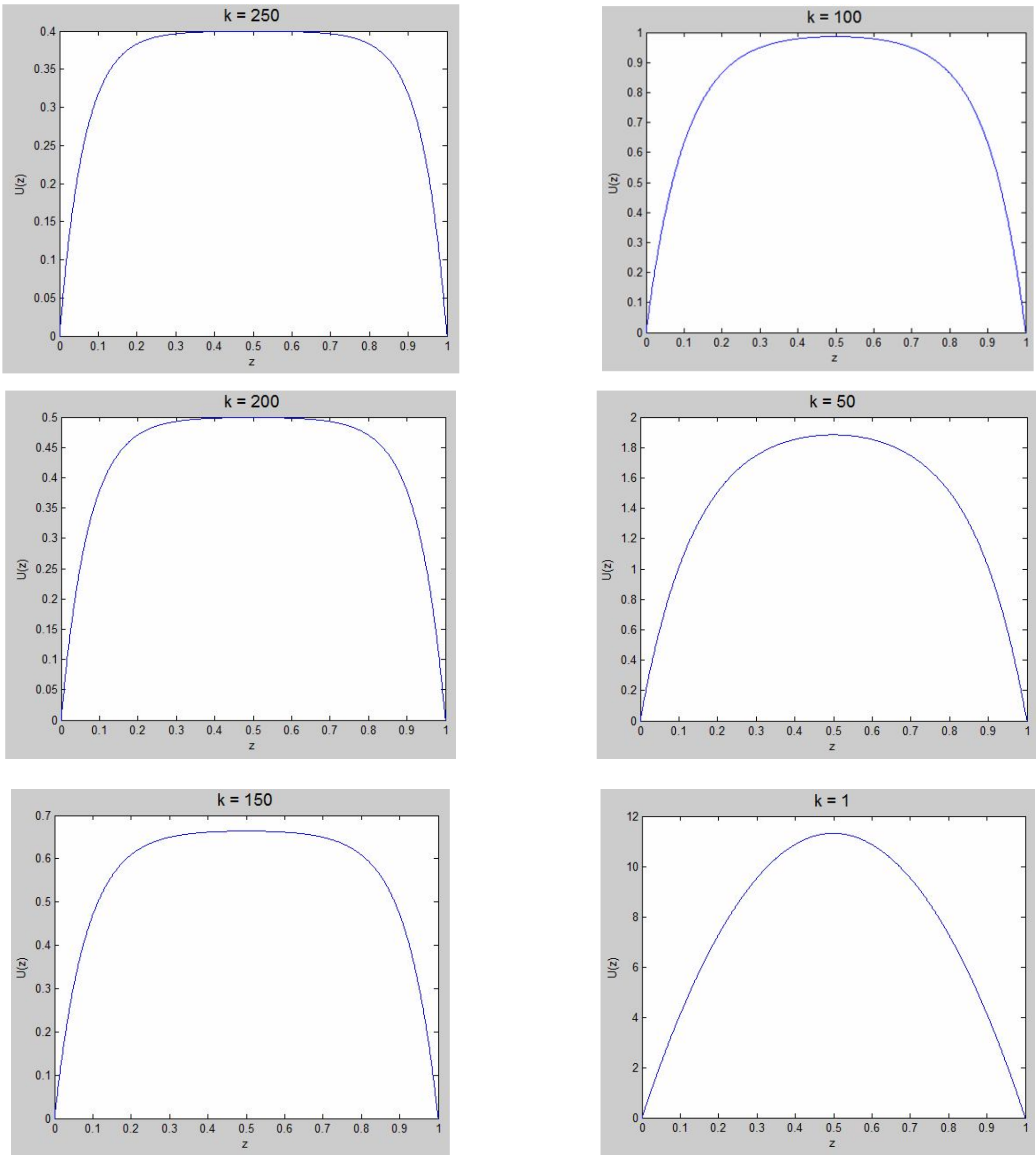

Figure 1. Stationary velocity profiles for turbulent flow between two flat-plates for $k = 1, 50, 100, 150, 200, 250$.

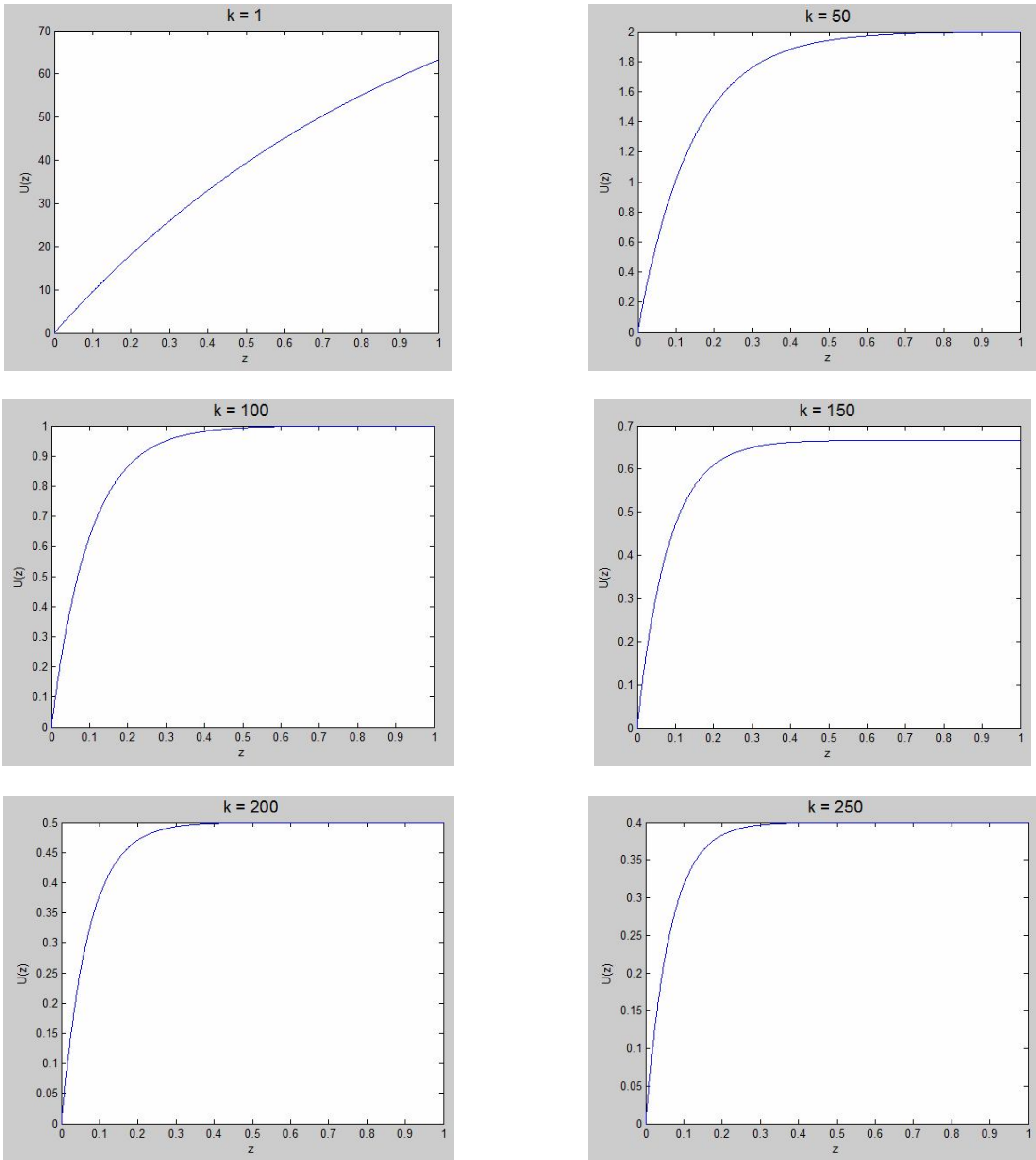

Figure 2. Wall Turbulence in a single plate for $k = 1, 50, 100, 150, 200, 250$.

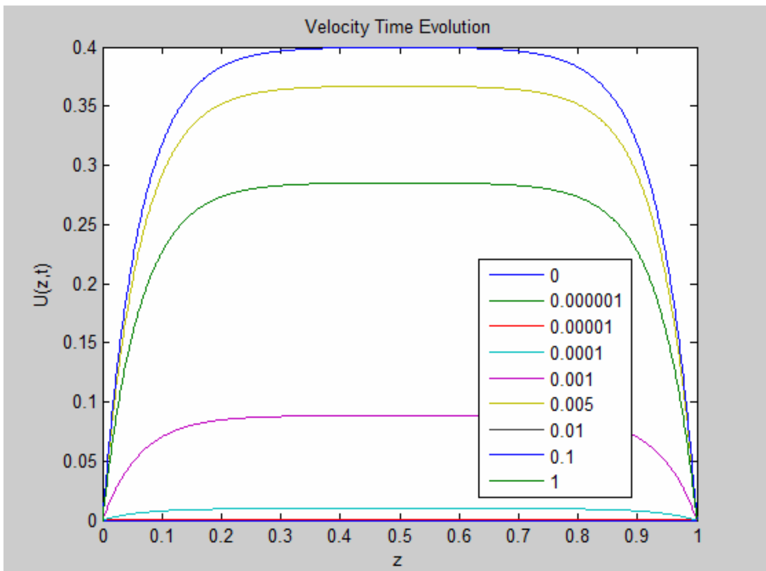

Figure 3. Velocity time evolution for turbulent flow in a two-plate system for various t values

3.3 Model fitness with experimental data

In Figure 4, we compared the fit of our equation with experimental data at low Reynolds numbers for turbulent flow in flat plates [16]. The exact data points used by Shiyi Chen et al. to validate the results of their viscous Camassa-Holm equations as a closure approximation for the Reynolds averaged equations of the incompressible Navier-Stokes fluid [15]. The velocity profile was plotted as well for the von Karman log law predictions using 0.41 and 5.5 for the von Karman constant and the additive constant, respectively [13].

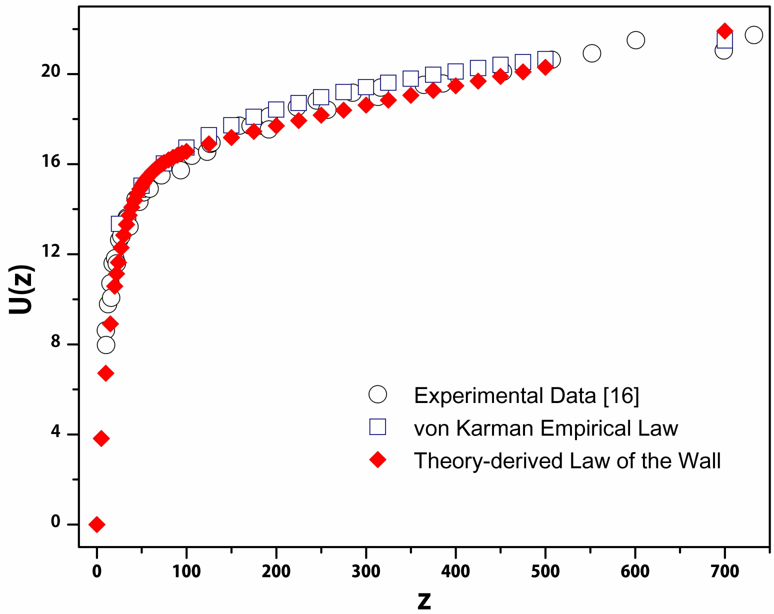

Figure 4. Mean Velocity Profile in the Channel for the new Law of the Wall and von Karman log law compared with the experimental data.

As the Reynolds numbers for these experiments are small, g, k, and L somehow have not reached their theoretical asymptotic values. Our new law of the wall used the abovementioned adjustable parameters for the fitting. We obtained the best data fit for the experimental data when we assumed that the probability parameter k is slowly changing with the wall distance z. By definition, k is proportional to the probability of kicking particles to different momentum. Hence, k can be a function of the wall distance due to the varying levels of turbulence along the wall. As seen in Figs. 1 and 2, the shape of the profile is greatly affected by the parameters k and g which for the optimized fitting were found to be $k(z) = 0.0001 - 0.000046\sqrt{z}$ and $g = 0.00522$. Parameter L has minimal effect on the velocity curves for values greater than 1000. This assumption is justified since it is well established that the fluid is more turbulent near the wall. To provide proper scaling of the experimental data, g has been chose to have a small value. It is also known that the von Karman log law fails in the regions near and far from the wall. The new exponential curve on the other hand explains the linear behaviour near the wall. As predicted from our equation, at the region very near the wall, equation 6 simplifies to a linear form, $U_w \cong gz/\sqrt{k}$. We emphasize here that our predictions are consistent with the von Karman and experimental data at a wide range of distances from the wall, as seen in Figure 4.

4. Conclusion

In the present paper, a new theory of the law of the wall, regarding the nature of wall-bounded flow turbulence, is outlined in brief. In particular, the the analytical solutions of the modified NSE in stationary and non-stationary conditions and the validity of the law derived is focused upon. A self-consistent theoretical formulation for the law is derived which matches the experimental data at the log layer. Upon examination of the resulting velocity profiles in a fully developed two-plate system flows indicates that it can be used to model channel flows. The present observations emphasize that the above results cannot be obtained from the classical NSE.

Further data must be used to test the new law of the wall equation developed from the modified NSE especially the data from the measurements done on the turbulent velocity profiles in channel and circular pipe flows for a wide range of Reynolds numbers. To get the solution of the Equation 3 in cylindrical coordinates it suffice to replace in flat plate non-stationary solution the hyperbolic sine functions by first modified Bessel function I_0. Qualitatively, similar result was obtained and tested against experimental data by Chen et al. [15]. Their rather complicated result for wall turbulence velocity profiles was in terms of hyperbolic cosine functions and some power law terms obtained from a closure of Reynolds-averaged equation (Camass Holm equations). For the turbulent pipe flow the cosh functions could be just replaced by first modified Bessel functions in agreement with our result except the power law terms [17]. We presented a more simple result. Also, the fundamental assumption requires further investigations, both experimentally and theoretically, i.e. on basis of the quantum nature of turbulence.

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Nomenclature

U	Mean velocity
v	Kinematic viscosity
u_*	Friction velocity
τ_W	Wall skin friction
ρ	Fluid density
K	von Karman constant
B_i	Additive constant in the von Karman logarithmic law
γ	Coefficient in Chorin and Barenblatt power law
σ	Probability per unit time that a particle of mass m is imparted a momentum kick Π
Π	Momentum kick
P_{ij}	Pressure tensor
σ_{ij}	Components of the shear stress
g	Constant effective pressure head per unit kinematic viscosity that drives the motion of the fluid

References

[1] Getreuer, P., A. Albano, A. Muriel. *Possibility of turbulence from a post-Navier-Stokes equation*. Physics Letters A, 2007. **366**(1-2): p. 101-104.

[2] Cipra, B., *A new theory of turbulence causes a stir among experts*. Science, 1996. **272**(5264): p. 951

[3] Aly, H.S. *Towards a Better Understanding of Turbulence.* CFD Letters, 2009. 1(1): p. 2-3

[4] Landau, L.D., and Lifschitz, E.M. *Hydrodynamics* (Nauka, Moscow, 1988).

[5] Fox, G. and Germano, F. *Fluid mechanics,* Van Nostrand Co. Inc., Princeton, USA. 1961.

[6] McComb, W.D. *The physics of fluid turbulence*. Oxford, Clarendon Press. 1992.

[7] Ruelle, D. *The Turbulent Flow as a Dynamical System, New Perspectives in Turbulence*, edited by L. Sirovich L. Springer, New York, USA. 1991

[8] Oberlack, M. *A unified approach for symmetries in plane parallel turbulent shear flows*. Journal of Fluid Mechanics, 2001. **427**(1): p. 299-328.

[9] George, W. K. *Is there a universal log law for turbulent wall-bounded flows?* Philosophical Transactions of the Royal Society A, Mathematical, Physical and Engineering Sciences. 2007, **365**(1852): p. 789-806.

[10] Guo, J. and P.Y. Julien. *Modified log-wake law for turbulent flow in smooth pipes*. Journal of Hydraulic Research, 2003. **41**(5): p. 493-501.

[11] Bradshaw, P. and G.P. Huang. *The Law of the Wall in Turbulent Flow*. Proceeding of the Royal Society of London A, 1995. **451**(1941): p. 165-188.

[12] Chorin, A., and G. Barenblatt. *Scaling of the intermediate region in wall bounded turbulence: The power law*, Phys. Fluids, 1998, **10**: p. 1043-1044.

[13] Zanoun, E., F. Durst, H. Nagib. *Evaluating the law of the wall in two-dimensional fully developed turbulent channel*. Phys. Fluids, 2003. **15**(10) 3079-89.

[14] Fox, G. and F. Germano. *Fluid mechanics*, Van Nostrand Co. Inc., Princeton, USA. 1961.

[15] Chen, S., C. Foias, C. Holm, E. Olson, E. Titi, S. Wynne. *Camassa Holm equations as a closure model for turbulent channel and pipe flow*. Physical Review Letters, 1998. **81**(24): p. 5338–5341

[16] Wei, T. and Willmarth, W.W. *Reynolds number effects on the structure of a turbulent channel flow*. J. Fluid Mechanics, 1989. **204**: p. 57-95.

[17] Abramowitz, M. and Stegun, I.A., Handbook of Mathematical Functions. Dover, New York, USA. 1972, 9th ed.